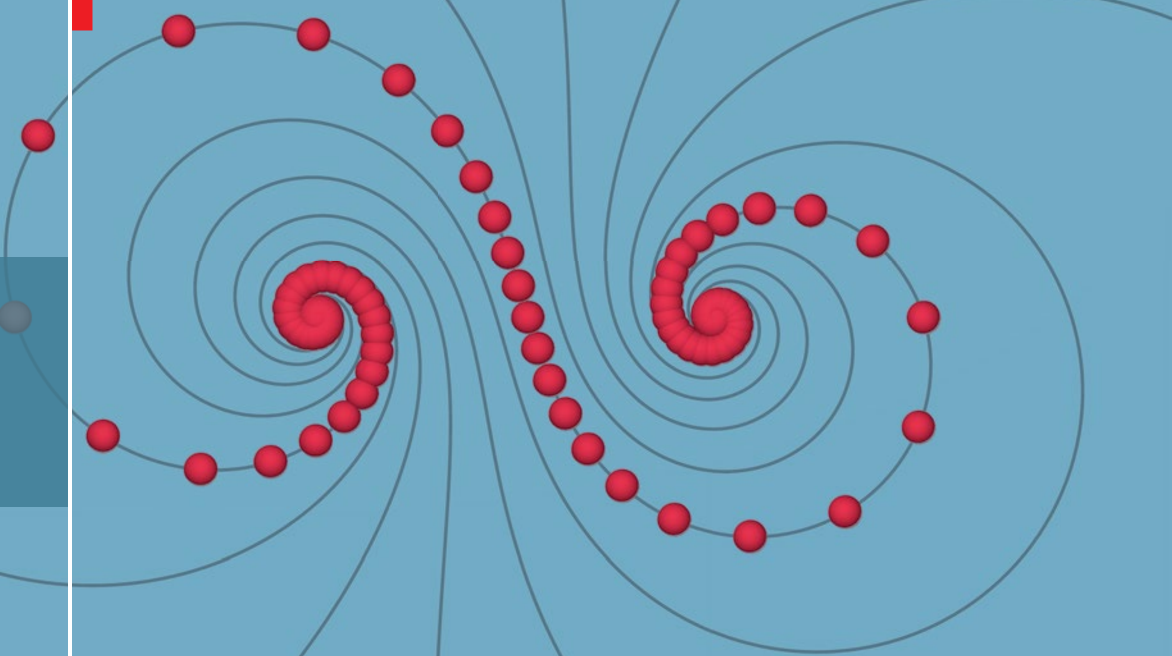
Misha Gromov



Great Circle of Mysteries

Mathematics, the World, the Mind

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Introduction

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Geometry is one and eternal shining in the mind of God.

JOHANNES KEPLER

God's plan for mathematics is beyond human reach but mathematics is the only light that can illuminate the mysteries of this world.

What we call the *science of physics* is filled with the radiance of mathematics; the fog of ignorance around what we now call LAWS OF NATURE receded in the brilliance of this physics.

But the beam of this light has barely touched the edges of the kingdom of LIFE and the face of PRINCESS THOUGHT remains hidden from us in the shroud of darkness.

And unless we know the ways of THOUGHT we cannot understand what mathematics is.

The first part of the book – *Quotations and Ideas* – is sprinkled with the ideas of those who saw sparks of light in the dark sea of the unknown. In the second part – *Memorandum Ergo* – we reflect on what in mathematics could shed light on MYSTERY THOUGHT.

Quotations and Ideas

1. Beautiful Elsewhere

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Round us, near us, in depth and height,
Soft as darkness and keen as light.

ALGERNON SWINBURNE, LOCH TORRIDON

"*Mathématiques, un dépaysement soudain,*" was an exhibition organized in 2011 by *Fondation Cartier pour l'art contemporain* in Paris. It featured, among other things, *The Library of Mysteries*:

THE MYSTERY OF PHYSICAL LAWS
THE MYSTERY OF LIFE
THE MYSTERY OF THE MIND
THE MYSTERY OF MATHEMATICS

These were presented as quotes from writings of great scientists in a film made by David Lynch – an artist's visualizations of the ideas of Time, Space, Matter, Life, Mind, Knowledge, and Mathematics.

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Michel Cassé and Hervé Chandés have persuaded me to try to do a *Naive Mathematician's* version of what David Lynch has done – to project a vision of these ideas to the invisible screen in our mind, an image illuminated by the eternal shining of mathematics rather than by the reflection of the beauty of the goddess of arts.

I knew it would be impossible but I tried anyway. Much of what I wrote was done in discussions with Giancarlo Lucchini and some of my English was corrected by Bronwyn Mahoney. Below is a modified version of what came out of it.

2. Science

Nothing exists except atoms in the void;
everything else is opinion.

Democritus of Abdera (?), 460–370 BCE

All men by nature desire knowledge. Thinking is the talking of the soul with itself. ... knowledge is the one motive attracting and supporting investigators ... always flying before them ... their sole torment and their sole happiness. To perceive is to suffer.

Common sense is the collection of prejudices acquired by age eighteen.

... Science ... [*is*] *asking, not whether a thing is good or bad ... but of what kind it is?*

Science is the belief in the ignorance of experts. What we know already ... often prevents us from learning. Our freedom to doubt was born out of a struggle against authority in the early days of science.

Science is no more a collection of facts than a heap of stones is a house. A fact is valuable only for the idea attached to it, or for the proof that it furnishes. He who does not know what he is looking for will not understand what he finds. The investigator should have a robust faith – and yet not believe.

Science increases our power in proportion as it lowers our pride. Ignorance more frequently begets confidence than does knowledge. Whoever undertakes to set himself up as a judge of Truth and Knowledge is shipwrecked by the laughter of the gods.

The gods are fond of a joke – the universe is not only queerer than we suppose, but queerer than we can suppose. And the most incomprehensible thing about the world is that it is comprehensible.

There is geometry in the humming of the strings, there is music in the spacing of the spheres. A hidden beauty is stronger that an obvious one. It is godlike ever to think on something beautiful and on something new.

The most beautiful thing we can experience is the mysterious. It is the source of ... art and ... science – branches of the same tree. He to whom this emotion is a stranger ... is as good as dead: His eyes are closed.

When it comes to atoms, language can be used only as in poetry. Poetry is nearer to vital truth than history. Knowledge is limited. Imagination encircles the world.

But put off your imagination, as you put off your overcoat, when you enter the laboratory. Put it on again when you leave.

The objective reality of things will be hidden from us forever; we can only know relations. Everything we call real is made of things that cannot be regarded as real. The internal harmony of the world is the only true objective reality.

It is not nature that imposes time and space upon us, it is we who impose them upon nature because we find them convenient. ... the distinction between past, present, and future is a most stubbornly persistent illusion.

A human being is a part of a whole, called by us "universe" – a part ... restricting us to our personal desires and to affection for a few persons nearest to us. Our task must be to free ourselves from this prison by widening our circle of compassion to embrace all living creatures and the whole of nature in its beauty.

Pythagoras 1	Charles Darwin	Niels Bohr
Heraclitus 1	Claude Bernard	Albert Einstein
Plato 1	James Clerk Maxwell	John Haldane
Aristotle 1	Henri Poincaré	Richard Feynman

What these people think and how they write enlightens your mind and elevates your spirit but these thoughts have little life of their own. They do not grow, they do not transform, they do not shoot new green sprouts – they luminesce as crystals of fiery flowers frozen in eternity. They are not quite what mathematicians call *ideas*, they are halfway between *ideas* and *opinions*.[1]

Great scientific *ideas* are different – they are alive, they ignite your soul with delight, they invite you to fight and to contradict them. Unlock your spirit from the cage of the mundane, let your imagination run free, start playing with such ideas as a little puppy with its toys – and you find yourself in the world of *Beautiful Elsewhere* – that is called *Mathematics*.

[1] An opinion about X is a function, say $OP_X = OP_X(p)$, that assigns yes (agree) or no (disagree) to a person p who is dying to say what he/she thinks about X; e.g., about the existence (non-existence) of *vacuum*. Democritus did not care about specific yes/no-values of $OP_X(p)$, except for p being his best friend, maybe. But the philosopher would love to see a correlation between $OP_X(p)$ for $X =$"*vacuum exists*" and the distance from the house of p to Abdera.

3. Numbers

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All the mathematical sciences are founded on relations between physical laws and laws of numbers.

James Clerk Maxwell (1856)

The numbers named by me exceed the mass equal in magnitude to the universe.

Archimedes, The Sand Reckoner, 250 BCE (?)

Archimède ... dont la vie est une époque dans l'histoire de l'homme, et dont l'existence parait un des bienfaits de la nature.

Nicolas Condorcet

Archimedes estimated the diameter of the universe at about 2 light-years that is $\approx 2 \cdot 10^{13}$ km – twenty thousand billions kilometers, about half, as we know today, of the distance to the nearest stars – the binary system Alpha Centauri A & B.

Then Archimedes invented an exponential representation of large numbers and evaluated the number of sand grains or rather of ≈ 0.2 millimeter poppy seeds needed to fill it at less than 10^{63} in modern notation. (I took these numbers from the Wikipedia article. In fact the $2 \cdot 10^{13}$ km cube has volume $8 \cdot 10^{57}\,\mathrm{mm}^3$ that makes 10^{60} cubes of 0.2 mm.)

If a philosopher would not be impressed and say that

a good decision is based on knowledge and not on numbers,

Archimedes might respond that decisions may be left to our mighty rulers but that *numbers* are the guardians of our true knowledge.

Large numbers are everywhere. Even Socrates, Plato, and Aristotle would admit that *knowledge of thyself* is incomplete if you are unaware of the roughly 10^{14} (100 000 billion) bacteria in your guts – several bacteria per each cell of your own body.

(Bacteria are roughly 1 μm – one thousandth of a millimeter = one millionth of meter – in size, a few thousand times smaller than your own cells volume-wise. If you had a bacterium inside every one of your cells you would hardly notice this – you will be safely dead by that time.)

A single bacterium, if there are enough nutrients, can divide every 20–30 minutes and, in 24 hours you may have a blob $10^5\mu$m = 10 cm in size with about $2^{50} = \left(2^{10}\right)^5 = 1024^5 \approx 1000^5 = 10^6 \times 10^9$ (million billions) bacteria in it.

A schoolboy now computes:

Day 2. The blob contains $10^{15} \times 10^{15} = 10^{30}$ bacteria and is $10^{10}\mu\mathrm{m} = 10\,\mathrm{km}$ in diameter, about 1 kg of bacteria per every square meter of the surface of the Earth.

Day 4. It increases to $10^{20}\mu\mathrm{m} = 10^{11}\,\mathrm{km}$ in diameter and reaches the outermost regions of the Solar system. It will engulf the Sun ($\approx 1.5 \times 10^8$ km from Earth), all planets including Pluto ($\approx 6 \times 10^9$ km) but not fully the orbit of Sedna at its farthest point from us ($\approx 1.4 \times 10^{11}$ km).

(Aristotle, who maintained that

the shape of the heaven is of necessity spherical,

would feel relieved if he knew that the bacteria are still contained by the *spherical heavenly shell* of the hypothetical *Oort cloud* of comets around the Sun about a light-year away.)

Day 7. The blob contains $10^{15\times 7} = 10^{105}$ bacteria and has $10^{5\times 7}\mu\mathrm{m} = 10^{26}\,\mathrm{km} \approx 10^{13}$ light-years in diameter, a hundred times the diameter of the observable universe ($\approx 10^{11}$ light-years).[2]

Bacteria have been around for billions of days, but do numbers like $10^{100\,000\ldots}$ make any sense at all? The answer is *yes and no*. They cannot be reached by counting 1, 2, 3, . . . , at least not in our space-time *continuum*, nor be represented by *collections of physical objects* of any kind. However, unrealistically large (and unrealistically small) numbers are instrumental in our treatment of NATURE'S LAWS that are manifested in observable properties of *objects* in the Universe.

How does Nature, who, as Einstein says, *integrates empirically,* manage to satisfy these laws?

Is it because she has something *much bigger* than space/time (kind of quantum fields?) at her disposal where *empirical integration* is possible?

Or is there a *secret logical something* built into Nature and she proceeds by *mathematical induction* as mathematicians do?

Or had she found a *simple logical bypass* for arriving at these laws but we cannot reach it being bound to the mental routes available to our brains?

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[2]Darwin computed the numbers of descendants of a couple of elephants and arrived at the number 15 000 000 after 500 years. This is much exaggerated, but in 5 000 years the Earth would be covered several times over by more than 10^{15} elephants and the whole Universe would not contain 10^{90} elephants after 30 000 years. (30 000 is yesterday on the Earth geological time scale, it is less than 0.02% of (about 200 000 000 years of) the evolution time of mammals on Earth.)

These questions, probably, make no sense; it is frustrating being unable to formulate a good one.

Yet, a mathematician may find a consolation in trying to estimate the number N_{can} of different logical arguments (brain routes), say in L words, that a sentient brain *can*, in principle, generate. Probably, if somebody told our mathematician what the words *can* and *in principle* signify, he/she would bound N_{can} by something like $\sim L\log L$, or even less than that – far from the number $N_{all} \sim 2^L$ of *all* such arguments, well behind what bacteria can do.

This might hurt his/her pride but then the mathematician will soon realize that numbers that linger behind his/her own logic/language beat the fastest replicating bacteria.

Indeed, *think* of Schrödinger's cat. The body of a cat is, roughly, composed of $N \approx 10^{26}$ molecules, those of water and small molecular residues in macromolecules. Suppose each molecule can be in two states. Then there are $S = 2^N \approx 10^{0.3N}$ *states of a cat*. Some of these states are judged being *alive* and some are classified as *dead*. The number, say $\mathcal{CAT}$, of possible judgements/opinions is

$$2^S = 2^{2^{10^{26}}} > 10^{10^{10^{25}}}$$

How does one decide, how can one select a sound judgement from this super-duper-universe of possibilities? Mathematicians do not understand how it works, but a cat, if he/she is alive and oblivious of math., somehow manages, makes a right choice and ... stays alive.

Some courageous people play with unimaginably greater numbers, the descendants of *Gödel's incompleteness theorem*. If you meet such a number on your path of reasoning about "real world" your logic is as good as dead.[3] Fortunately, you do not meet them in "real life" unless you call these misshapen monsters by their names.

THE MONSTER OF **STOP**. If your computer has M bits of memory, say with $M = 10^{10}$, then whatever you "ask" the computer to do, it either *stops* after $< 2^M$ steps or it goes into a cycle and runs forever. (You can use convenient *time units* instead of "steps"; the number $2^{10^{10}}$ is so big, it makes little difference if you take nanoseconds or billions of years for these units.)

Here, what we call a *question* or a *program* that you "ask" your computer to perform, is a sequence of *letters* that are the names of keys on the keyboard you have to press in order to activate this *program*.

Let us leave the "real world" and allow your computer to have an *infinite* (unbounded) memory. As in the finite case, the computer may stop after finitely many steps or run for infinite time *depending on your question*, (and on the design of the hardware and the operating system in your computer) except that *infinite* does not have to be *cyclic* for infinite memory. For example, if asked to find a file

[3] These numbers lie at the heart of *Turing's halting theorem* and of Kolmogorov–Chaitin complexity. A whiff of either of the two is poisonous for any *scientific* theory.

named *cell-phan-nimber-Bull-Gytes* the computer either finds it and stops, or, if there is no such file, runs forever.

(A fundamental ability of your *slow* brain, not shared by the *fast* Windows search system, is an *almost instantaneous* NO SUCH FILE response to this kind of inquiry. A simple, yet structurally nontrivial, instance of this is BITTER on your tongue for almost anything chemically away from potential nutrients, with a few mistakes; e.g., SWEET for saccharin.

One may speculate together with Robert Hooke, Charles Babbage, and Alan Turing on plausible brain memory architectures that makes this possible. There is no experimental means for checking non-trivial conjectures of this kind but there is a mathematically attractive model of such a memory suggested by Pentti Kanerva, called *sparse distributed memory*.)

Pick up a (moderately large) number L and take *all those programs* in L letters for which the computer *eventually stops*. Since the number of these programs is finite ($< 100^L$ if there are < 100 different letters at your disposal for writing programs) the longest (yet, finite!) time of executing such a program, say, the time measured in years, is also *finite*; call this time **STOP**(L).

Although finite, this number of years, even for moderate L, say **STOP** $=$ **STOP**(50 000), (a program in 50 000 letters takes a dozen pages to write down) is virtually indistinguishable from infinity in a certain precise logical sense.

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Our Universe is pathetically small not only for containing anything of this size but even for holding a *writing* of an *explicit formula* or of an *explicit verbal description* of a number of such size, even if we use atoms for letters in such a writing. (The way we have just described this number does not count – distinguishing stopping times from infinity without indicating a specific "experimental protocol" for this is not what we call *explicit*.)

By comparison, $\mathcal{CAT}$ appears tiny – the corresponding exponential formula can be expressed with a few dozen binary symbols (and *physically* written down with a few thousand atoms manipulated by means of an *atomic force microscope*).

To be fully honest we must admit there was something missing from our definition of **STOP**(L).

Imagine, for example, that the memory of our computer does contain the string *cell-phan-nimber-Bull-Gytes* but it is positioned so far away that it cannot be reached in less than T time units. Since you can choose T as large as you wish, the logic of our definition inevitably makes **STOP**(L) equal infinity, even if you limit the length L of admissible programs to something quite small, say, less than one thousand letters.

Somehow one has to prohibit such a possibility, by insisting that all "memory cells" in you computer that *cannot* be reached in less than, say, 10^{10} time units are empty, nothing is written in these "far away cells". Moreover, the computer is supposed to know when it crosses the boundary of "non-empty space" and will not spend any time in searching the empty one.

On the other hand, the computer is allowed, in the course of a computation, to write/erase in these "far away cells"; this is what may eventually generate an enormous volume of occupied memory cells and make its reading excruciatingly long.

With this provision, the definition of **STOP**(L) becomes correct, it does give you something *finite*, PROVIDED you have *precisely* defined what "far away cells" and "reaching something in the memory" mean.

But can one *explicitly* describe in *finitely many* words an *infinite* memory along with a description of a search program through this memory?

A commonly accepted solution articulated by Turing, is to assign memory cells/units to *all* numbers $1, 2, 3, 4, \ldots, 1000\ldots$ with labels "*empty*"/"*non-empty*" attached to them, and with the symbols 1 and 0 written along with all "*non-empty*" marks. Then an individual step of a memory search is defined as moving from an i-cell to $i+1$ or $i-1$, where each cell is labeled "*non-empty*" after having been visited.

If you are susceptible to the magic of the word "all" and believe this truly defines the infinity of numbers, then you will have a "mathematically precise" definition of **STOP**(L).[4]

All this being said, there is something not quite right with **STOP**: it is a byproduct of the machinery of formal logic – not a *true mathematical* number. But about thirty years ago, **pretty** $\mathcal{HUGE}$ numbers were discovered; e.g., the time Hercules needs to slay Hydra in a mathematical model of this regrettable parricide. But all such $\mathcal{GIANT}$ $\mathcal{BEAUTIES}$ are incomparably smaller than the ugly looking **STOP**.

[4]There are, probably, 5–10 mathematicians and logicians in the world who think that the monsters like **STOP** are indicators of fundamental flaws in our concepts: "number", "finite", "infinite".

4. Laws

Lex 1: *Corpus omne perseverare in statu suo quiescendi
vel movendi uniformiter in directum, nisi quatenus
a viribus impressis cogitur statum illum mutare.*

Isaac Newton, Philosophiæ Naturalis
Principia Mathematica, 1687[5]

*... les lois générales, connues ou ignorées,
qui règlent les phénomènes de l'univers,
sont nécessaires et constantes.*

Nicolas Condorcet

So much as we know of them [the laws of nature] *has been
developed by the successive energies of the highest intellects.*

Michael Faraday

The most astounding single event in the history of science, a physicist argued, was the discovery of *general relativity* in 1916 – this would have been postponed by 20–30 years if not for Einstein, twice as long as any other scientific discovery, according to our physicist's calculation on the back of an envelope.

Our physicist, a theoretician, could not confirm his calculation by an experiment, but history has done it for him and ... proved him wrong.

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The true logic of this world is the calculus of probabilities.

James Clerk Maxwell

[5]Historically, the *first recorded* LAW – the velocity/force formula for *motion in a viscous medium* – was stated by Aristotle. (Later versions of this were suggested by Newton, 1687, and by Stokes, 1851.) A century after Aristotle, Archimedes discovered basic laws of *statics of mechanical systems*: The laws of the lever and of equilibrium of solid bodies in liquids.

Mendel's theory of genes – units of inheritance – was published in 1866, where Mendel derived their very existence and essential properties from

the striking regularity with which the same hybrid forms always reappeared

in thousands of his experiments with pea plants.

The discovery of genes was the greatest event in biology since the discovery of *cells* by Robert Hooke in 1665 and of *infusoria* by Antonie van Leeuwenhoek in 1674.

Mendel's methodology of

combinatorial design of multistage interactive experiments

+

extracting specific structural information from statistics of observable data by mathematical means

was novel for all of science. This is why Mendel's paper was ignored by biologists for about thirty years.

When similar data were obtained and analyzed by de Vries, Correns, and Tschermak at the turn of the 20th century, biologists returned to Mendel's paper. Many, including Alfred Russel Wallace, were appalled by Mendel's ideas, while even the most sympathetic ones were confounded by Mendel's "counterintuitive" and "biologically unfeasible" algebra.

In 1908 a leading English mathematician G.H. Hardy and independently a German physician Wilhelm Weinberg spelled out this *counterintuitive* as

$$\frac{\left[(p+q)^2+(p+q)(q+r)\right]^2}{[(p+q)^2+(p+q)(q+r)]\cdot[(p+q)(q+r)+(q+r)^2]}=\frac{(p+q)^2}{(p+q)\cdot(q+r)}$$

and MENDEL'S LAWS OF INHERITANCE were accepted by (almost) everybody.

(Hardy called this *a mathematics of the multiplication table type*. He overlooked the mathematical beauty of *Mendelian dynamics* of the *next generation maps* M in the spaces of *truncated polynomials* – the maps that represent transformations of *distributions of alleles* in populations under *random matings*. In a simple instance, such an M applies to matrices $P=(p_{ij})$ by substituting each (i,j)-entry p_{ij} by the product of the sums of the entries in the i-row and the j-column in P,

$$(p_{ij})=P\overset{M}{\mapsto}P^{\text{next}}=(p_{ij}^{\text{next}})\text{ for }p_{ij}^{\text{next}}=\sum_i p_{ij}\cdot\sum_j p_{ij}.$$

It is amazing, albeit obvious, that $M(M(P))=\text{const}\cdot M(P)$ for $\text{const}=\sum_{ij}p_{ij}$, where this amounts to the above (p,q,r)-formula for symmetric 2×2 matrices in Hardy's notation.)

> *... are we justified in regarding them* [genes] *as material units; as chemical bodies of a higher order than molecules? ... It makes no difference in ... genetics. Between the characters that are used by the geneticist*

and the genes his theory postulates lies ... embryonic development.

Thomas Hunt Morgan, 1934

In 1913, almost half a century after the publication of Mendel's paper *Versuche über Pflanzen-Hybriden*, 21-year-old Alfred Sturtevant made the next step along Mendelian lines of logic and determined *relative positions* of certain genes on one of the chromosomes of Drosophila by analyzing frequencies of specific morphologies in generations of suitably interbred flies.

Just think about it. You breed fruit flies, you count how many have particular *combinations* of certain features; i.e., you record the distribution of occurrences of the following eight $(2 \times 2 \times 2)$-possibilities:

[striped bodies]/[*yellow* bodies]
[red eyes]/[*white* eyes]
[normal wings]/[*smallish* wings]

Then you distinctly see with your mathematically focused mind's eye (even if you happened to be as color blind as Sturtevant) that the corresponding genes – abstract entities of Mendel's theory – that are associated with these features, are all lying in definite relative positions on an imaginary line, where, as for Mendel's genes, the existence of this line is derived from how *hybrid forms reappear*. In particular, you assign the "eyes gene" a position *between* the "body gene" and the "wing gene", since

[smallish wings] + [yellow bodies] "**implies**" [white eyes]

in descendants of certain parents with an abnormally high probability.

Decades later, molecular biology and sequencing technology demystified "Sturtevant's line" by identifying it with a DNA string segmented by genes, but a mathematical unfolding of Sturtevant's idea still can be seen only in dreams.

About Robert Hooke, Antonie van Leeuwenhoek, Drosophila Melanogaster and the Idea of Sturtevant.

Hooke's name is associated with *Hooke's elasticity law* but that was only one of many of his experimental discoveries, original concepts, and practical inventions. For example, he recognized fossils as the remains of *extinct species*, he developed an almost modern *model of memory*, he proposed (1665?) the construction of *the spring balance watch* (Huygens' description of his own construction dates to 1675) and he suggested (1684) a detailed design of an *optical telegraph with semaphores.* (The first operational system with a network of ≈ 500 stations was built in 1792 in France.)

Leeuwenhoek found out how to obtain small glass balls for the lenses of his microscopes but he made others to believe he was grinding tiny lenses day and night by hand. Besides infusoria, he observed and described crescent-shaped bacteria (*large Selenomonads*), cell vacuoles, and spermatozoa. The secret of Leeuwenhoek's microscopes was rediscovered in 1957.

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Drosophila melanogaster – the pomace fly ($\approx 2.5\,\mathrm{mm}$ long, normally with red-eyes and stripped bellies) – were introduced as a major model organism in genetics by Thomas Hunt Morgan. He and his students were counting the mutant characteristics of thousands of flies and studied their inheritance. Analysing these data, Morgan demonstrated that genes are carried on chromosomes; also he introduced the concepts of *genetic linkage* and of *crossing over*.

Sturtevant mathematically "synthesised" his string of genes from the "substrate" of results and ideas taken from the work of Morgan almost as Kepler "crystallized" elliptical orbits from Tycho Brahe's astronomical tables. Sturtevant recalls the great moment as follows.

> *I suddenly realized that the variations in strength of linkage, already attributed by Morgan to differences in the spatial separation of the genes, offered the possibility of determining sequences in the linear dimension of a chromosome. I went home and spent most of the night (to the neglect of my undergraduate homework) in producing the first chromosome map, which included the sex-linked genes y, w, v, m, and r, in the order and approximately the relative spacing that they still appear on the standard maps.*

Sturtevant's idea of (re)construction of the (a posteriori linear) geometry of the genome is similar to Poincaré's suggestion for how the brain (re)constructs the (a posteriori Euclidean) geometry of the external world from a set of samples of retinal images.

Grossly oversimplifying, an unknown geometric (or non-geometric) structure S from a given class $\mathcal{S}$ on a set X under study – be it the set of (types of) genes in the genomes (of organisms) of a given species or the set of photoreceptor cells

in the retina – is represented by some probability measure on the set of subsets Y of X. What is essential, this measure is supported on $\mathcal{S}$-*simple* (*special*) *subsets* Y that admit *short descriptions in the language of* $\mathcal{S}$; this allows a reconstruction of S from relatively few samples.

This sampling is far from random. In genetics, such a $Y = Y(O)$ is the subset of genes of particular allele versions in the genome of an individual organism O, where these O are obtained via a *controlled* breeding protocol that had been specifically designed by an experimenter.

In vision, such a $Y = Y(t)$ is the set of excited photoreceptor cells in the retina of your eye at a given moment t, where more often than not the variation $Y(t)$ with t is due to a motion of an object relative to your eye; the brain, which commands the muscles that move your eye, has an ability to design/control such variations.

The hardest step in finding S is *guessing* what $\mathcal{S}$ is. After all, what is *a structure*?

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Mendel's laws are no more than a Platonic shadow – a statistically averaged image of the workshop of Life on the flat screen of numbers. The molecular edifice of the cell is crudely smashed on this screen and its exquisite structure cannot be reconstructed from this image by pure thought. Hundreds (thousands?) of ingenious experiments are needed to recover the enormousness of information that had been lost.

> *... I hold it true that pure thought can grasp reality,*
> *as the ancients dreamed.* ALBERT EINSTEIN

Contrary to what we see in biology, the mathematical image of the basic machinery running the world of physics retains the finest details of this machinery. It even may seem, probably, only to our *Naive Mathematician*, that the less you know the better you understand how the Universe is run.

For example, forget about velocities, forces, accelerations. Imagine a world exclusively populated by wandering watches that have no perception of speed and force. But when two watches meet, they can recognize each other and compare their records of the intervals of times between consecutive meetings.

A watch-mathematician would sum up what he/she believes he/she "sees" in the watch-world as *self-obvious axioms* and, after pondering for a few centuries on what they imply, he/she will figure out that there is a unique simplest most symmetric *watch space* of every given dimension. It is the *Lorentz–Minkowski time-space*, that is four-dimensional in the Universe in which we happened to exist.

The mathematician will be delighted by this marvelous space-idea; yet perplexed, since his/her mental picture of the world does not explain

> *why the watches that have no physical contact apart from their meeting points remain synchronous.*

(Here, on Earth, it is not this incredible synchronisation but its violation that is regarded as paradoxical.)

But then his friend physicist conceives the idea of speed and his colleague experimentalist designs fast traveling watches. The mathematician sighs with relief: the formulas of his/her theory (that is called *special relativity* on Earth), are perfectly right and desynchronisation is clearly seen for watches traveling with mutual relative speeds close to 1. (On Earth, this 1, that is the speed of light, is elegantly expressed as 299 792 458 ... × another unit of speed the meaning of which no watch-mathematician has ever been able to grasp.)

> *Lines of force convey a far better and purer idea ...*
>
> Michael Faraday, 1833[6]

> *Thy reign, O force! is over. Now no more*
> *Heed we thine action;*
> *Repulsion leaves us where we were before,*
> *So does attraction.*
>
> James Clerk Maxwell, 1876

> *The theory of relativity by Einstein ...*
> *cannot but be regarded as a magnificent work of art.*
>
> Ernest Rutherford[7]

[6] *Faraday must have grasped with unerring instinct ... spatial states, today called fields, ...*
Albert Einstein, 1940

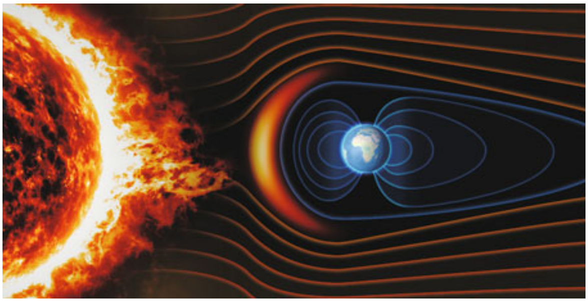
[7] Rutherford, named the Faraday of nuclear physics, experimentally identified alpha, beta and gamma emissions, discovered atomic nuclei and proposed protons and "nuclear electrons" for their constituents. (Neutrons were discovered and fit into the model of the nucleus about 20 years later.) *"All science is either physics or stamp collecting"* is a saying attributed to him. He, apparently, regarded chemistry as a part of physics and he did not live to see the birth of molecular biology from the seed of the Mendelian-chromosome theory of heredity.

Next our mathematician will look not for a most symmetric space but for the most symmetric *law* of "motion" applicable to all imaginable to him/her watch-spaces filled/strained by *lines/fields of force* à la Faraday.

First he/she thinks that no such distinguished law is possible, but then some terms in his/her calculation miraculously cancel each other and a beautiful equation pops out. He/she, undoubtedly, will call it the *Einstein vacuum equation*. (This was so derived by David Hilbert.)[8]

Then his friend physicist will bring in *energy/matter* and, to everybody's satisfaction, an experimentalist/cosmologist will show that the universe behaves as predicted by the resulting equation – the simplest mathematically imaginable description of [watch/space]-worlds – *general relativity theory*.

> *The greatest change in ... our conception of the structure of reality – since Newton ... was brought about by Faraday's and Maxwell's work on electromagnetic field phenomena.*
>
> ALBERT EINSTEIN, 1931



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> *... ten thousand years from now,*
> *the most significant event of the 19th century*
> *will be judged as Maxwell's discovery of the laws of electrodynamics.*
>
> RICHARD FEYNMAN, LECTURES ON PHYSICS, 1964.[9]

The *watch world* symmetry, that was acclaimed by the terrestrial physicists of the 20th century, originated in the work by Maxwell who, over 1855–1873, thought out (a system of twenty) differential (wave) equations that logically embrace and unify the *Ampere law* (1826) on magnetic effects of electric currents and *Faraday's law of induction* of electric currents by moving magnetic fields (1831).

---

[8]Hilbert was the one who, along with Poincaré, guided the passage from the 19th to 20th century mathematics.

[9]The 120th century seems far off – *another age must be the judge* – Charles Babbage's writes in his *Ninth Bridgewater Treatise* (1837). His design of *analytic engine* (that implemented what is now called the *universal Turing machine*) will be the most likely number one on the 19TH CENTURY SIGNIFICANT EVENTS list of a robot of the 22nd century.

Formally, these are *Lagrange–Hamilton equations* dictated by *Maupertuis' principle* for electromagnetic fields; they possess a remarkably high – what is now called *Lorentzian* – symmetry that, as Einstein wrote in 1953,

transcended its connection with Maxwell's equations.

(The historically first *wave equation* – the equation of a vibrating string, was written down and studied by d'Alembert in 1747; apparently, its *Lorentzian symmetry* went unnoticed until much later.)

The prima donna role for this symmetry in physics hinted at by FitzGerald in 1889 and suggested by Larmor in 1897 and by Lorentz in 1899, was consummated in two 1905 papers:

Sur la dynamique de l'électron[10] by Poincaré

and

Zur Elektrodynamik bewegter Körper by Einstein.

Two years later, Hermann Minkowski, who was usually preoccupied (besides *convex bodies*) with multidimensional geometry of *quadratic equations in many variables* (quadratic forms) and with their transformation groups, suggested a four-dimensional geometric realization of the (Poincaré)-Lorentz symmetry. The Minkowski 4-D geometry was extended by Einstein in 1916 to (*Einstein*)-*Lorentz spaces* that he used as a mathematical framework for *general relativity theory*.

Now, imagine a technologically backward watch-civilization where no high energy experiment is available. Here a mathematician will have to invent some mystical *absolute time* that will synchronize non-interacting watches as windowless *monads* of Leibniz.

Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration. ISAAC NEWTON

Then mathematician's friend physicist breaks a window somewhere from where he could see motions of other watches and the mathematician will realize that granting an *absolute time* to a Lorentz space necessarily implies an *absolute space* where motion is possible.

Eventually the two – the mathematician with his physicist friend – such a couple is called *Isaac Newton* on Earth – will arrive at the THREE LAWS and, when they come to Earth, they incorporate INVERSE SQUARE LAW for gravitation into their theory, where "square" is suggested (implied?) by the geometric

[10] where the *Lorentz group* materialized in the words

... the equations of the electromagnetic field are not altered by a certain transformation (which I will call by the name of Lorentz) of the form: ...

The *principle of relative motion* was discussed by Poincaré in earlier papers but that was not quite what we call *special relativity theory* as formulated by Einstein.

AREA $\sim R^2$-LAW: the area of the surface of the R-sphere is proportional to R^2 where the exponent 2 comes as $2 = 3 - 1$ for 3 being the dimension of our physical space.

(Newton mathematically *proved* that *only* the gravitation $\sim 1/R^2$-LAW agrees with astronomical observations,[11] but it is unclear who was the first to *conjecture* $\sim 1/R^2$. Robert Hooke claimed to be the one. Indeed, in his communication to the Royal Society of 1666 he states:

> *... heavenly bodies ... mutually attract each other within their spheres of action. ... so much the greater as the bodies are nearer.*

But others, including Galileo, Kepler, and Newton himself, expressed similar ideas on gravitation.

Newton regarded the proof of $\sim 1/R^2$ as by far more difficult than guessing it. Since he was the only person on Earth, if not in all of the astronomically observable[12] Universe, qualified to approach the problem and to evaluate its mathematical difficulty, we accept his judgement.)

But despite the remarkable agreement of the theory with recorded planetary orbits over short time intervals (thousands of years) a disturbing question will nag our friends.

Are these laws – the laws of classical mechanics + the inverse square law of gravity – *consistent* with the apparent stability of the solar system on the time scale of millions and hundreds of millions of years?

Newton himself believed the answer is NO and that planets would collide with the Sun if not for divine intervention from time to time. But about 250 years after Newton an optimistic *maybe* was suggested by a counterintuitive mathematical theorem, usually referred to as *KAM*, that (very roughly) says that *quite a few* physically significant dynamical systems may behave "asymptotically rather periodically", contrary to what physicists (as well as mathematicians) had always believed. (Everybody's intuition suggested "asymptotically chaotic" behavior of the *predominant majority* of mechanical systems with more than two degrees of freedom.)

This theorem depends on a hidden *symplectic symmetry* between the kinetic and potential energies as seen on yet another mathematical screen discovered by Hamilton ≈ 100 years after the death of Newton.

[11] Planetary motions to which Newton's equations apply are *not* directly observable. It took the *successive energies of the highest intellects* of (not only) Copernicus, Tycho Brahe, and Kepler to make the observable data amenable to mathematical analysis.

[12] "Observable" here, rather parochially, refers to an observer centred somewhere in the vicinity of *the Milky Way galaxy*.

Quantum electrodynamics describes Nature as absurd
from the point of view of common sense.
And it fully agrees with experiment.

Richard Feynman, QED: The Strange Theory of Light and Matter, 1985

Nothing is too wonderful to be true,
if it be consistent with the laws of nature;
and in such things as these experiment
is the best test of such consistency.

Michael Faraday,
Laboratory journal entry, 1849

Is the running of "watches" themselves governed by classical/relativistic mechanics?

Probably it is not hard to "prove" by an argument in the spirit of *Zeno's paradoxes* that no mechanical/electromagnetic model of Newton+Maxwell+Einstein can be compatible with the properties of *matter* we see everywhere around us. Apparently, our beautiful physical laws are "just" Mendelian kinds of images of something else, where this "something else" is expected by physicists to live in the *quantum world*.

Mathematical fragments of *quantum* may be accessible to us but when we try to imagine it as a whole our mind revolts, and if we insist it becomes giddy in a tangle of paradoxes and ambiguities. And little consolation can be found in what physicists keep repeating after Niels Bohr:[13]

If anybody says he can think about quantum physics without getting giddy, that only shows he has not understood the first thing about them.

You can hardly see a clear picture of anything while being giddy, but you would feel better if you could prove mathematically that, in principle, no common sense (including rigorous mathematics) model of *anything* resembling the *physical world* is possible.

"Thinking quantum" is incompatible with our deeply ingrained intuition of "reality". As Wolfgang Pauli[14] says:

The layman always means, when he says "reality" that he is speaking of something self-evidently known; whereas to me it seems the most important and exceedingly difficult task of our time is to work on the construction of a new idea of reality.

He also writes:

[13]In 1913, Bohr invented the concept of *quantization of a physical system* in the context of the *planetary model of the atom* that was suggested by Rutherford in 1911.

[14]Pauli (1900–1958) introduced *spin* into quantum mechanics, formulated the *Pauli exclusion principle*, and he predicted the existence of *neutrinos*.

Just as in the theory of relativity a group of mathematical transformations connects all possible coordinate systems, so in quantum mechanics a group of mathematical transformations connects the possible experimental arrangements.

This is only a hint about reality being a property of some transformation group, but a realistic idea of "physical reality" remains beyond our wildest dreams.

And quoting Pauli again,

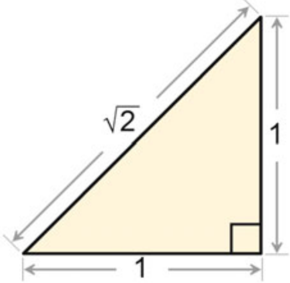
The best that most of us can hope to achieve in physics is simply to misunderstand at a deeper level.

Yet, maybe, there remains a hope of seeing the image of the world in the mirror of mathematics.

ἀγεωμέτρητος μηδεὶς εἰσίτω.
Let no one ignorant of geometry enter.

This is associated with the name of Plato who did not think much of those unfortunates who were

ignorant that the diagonal of a square is incommensurate with its side.



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Twenty four centuries later, certain of Plato's disciples went further by suggesting that *true human consciousness* is distinguished by the ability of its possessor to grasp validity, meaning and implications of "incommensurability" between certain *theories* of numbers (rather than between certain individual numbers) that follows from *Gödel's incompleteness theorem*.

According to this supposition, any conceivable *imitation* of human mind, be it by an electronic implementation of a Turing universal computer or by a bio-robot, would be instantaneously recognizable by the flagrant absence of this ability.

However, no significant data on a presence/absence of this ability in human populations have been collected despite the renewed interests of neuropsychologists in the problem of consciousness.)

Whoever despises the high wisdom of mathematics
nourishes himself on delusion.

Leonardo da Vinci

Every new body of discovery is mathematical in form,
because there is no other guidance we can have.

Charles Darwin

To those who do not know mathematics
it is difficult to get across a real feeling
as to the beauty, the deepest beauty, of nature.

Richard Feynman

When does a fragment of science qualify for a Law of Nature? Why does finding these laws require *successive energies of the highest intellects*?

Isn't a law *just* a *compression of information*, a *record of systematic correlations* between *facts* where these correlations can be found by *analyzing* your *observations*?

A "Yes" or "No" answer to this question depends on how you understand *just*, *information*, *systematic*, etc., where an essential (but not the only) difficulty facing a "law maker" is seen in the following example.

Imagine, that your *facts* or *events* are numbers, where the ones you happened to observe are

$$7,\ 19,\ 37,\ 56,\ 61,\ 91,\ 127,\ 189,\ 208,\ 296,\ 342,\ 386.$$

The only effective approach to finding "the law" behind these numbers is ... to *guess*: these numbers are differences of cubes[15],

$$1 = 1^3, \quad 8 = 2^3, \quad 27 = 3^3, \quad 64 = 4^3, \quad 125 = 5^3, \quad 216 = 6^3, \quad 343 = 7^3, \ldots.$$

For instance,

$$91 = 216 - 125, \quad 127 = 343 - 216, \quad 189 = 216 - 27, \quad 386 = 9^3 - 7^3.$$

(This is kind of how it happened in Bohr's rendition of the *Balmer–Rydberg formula* for the wavelength λ of the Hydrogen spectrum, e.g., observed in the stars. This formula, in suitable units, says that $\frac{1}{\lambda} = \frac{1}{m^2} - \frac{1}{n^2}$, with the rationale behind this provided by Bohr's quantized model of the Hydrogen atom.)

In general, a mathematician may say that a law $\mathcal{L}$ is a "simple" map (function) from some "small and simple" *parameter set* P, (e.g., the set of pairs of cubes of integer numbers) *onto* a "complicated" subset Obs of observable events (differences of cubes of integers) within another "simple but possibly large" set Ima of all imaginable events (e.g., of all integers or, if you wish, of all real numbers that make a much larger, yet logically equally simple set).

[15] *A little mathematics of the multiplication table type* tells you that the predominant majority of numbers are *not* differences of cubes. This makes this $(m^3 - n^3)$-*law* quite restrictive; hence, informative.

The number of conceivable *simple laws* $\mathcal{L}$, e.g., of those describable in a few dozen words, is enormous – it grows exponentially with the number of words/symbols encoding an $\mathcal{L}$. There is no (?) general rule for guessing such an $\mathcal{L}$, from a realistic number of samples from *Obs* in each particular case.[16] And *he who does not know what he is looking for will not understand what he finds* – as Claude Bernard says.

(One may object by pointing out that an animal/human brain builds up a coherent mental picture of the external world by *systematically* finding correlations and cutting off redundancies in the flows of electric/chemical signals it receives from other parts of the body.

It is hard to argue since we do not know what is the *system* used by the brain, but, in any case, this system was designed by Nature *not at all* for discovering her inner workings.)

If you are a biochemist rather than a mathematician, you may visualize *a Law of Nature* as an intricately shaped "molecule", a kind of "logical enzyme" for "catalytic synthesis of theories" out of "solution of empirical facts", where the role of solvent – the supporting matrix of the process – is played by mathematics.

The *architecture of logic* may vary from law to law and the corresponding *chemistries of facts* may have nothing in common (as statistical mechanics and classical genetics, for example) but the principles of *mathematical catalysis* remain the same for most (all?) *enzymatic laws* worthy of the name LAW OF NATURE.

But beware: A shorthand expression of a law, e.g., of inertia – *Corpus omne perseverare ...* – a string of a hundred characters, tells you as little of the structure and the function of a law as a sequence of hundred amino acids says by itself about the physiological role of the corresponding protein. In order to read the message encoded by such a string you need to have a fair insight on the mathematical nature of the *solvent matrix* as much as on the *natural chemistry of facts*.

And if no mathematics in your head is available for this purpose, you shall not understand much of the corresponding law if at all.[17]

This was what happened to Mendel's ideas – apparently, mathematics and physics were not in the biology curriculum of that time. If Mendel, instead of Nägeli, had corresponded with people like Boltzmann, Guldberg and Waage, or van 't Hoff, the timetable of genetics might have been shifted by a quarter of a century.

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[16] *Making* a correct guess is, in most cases, much harder than *verifying* one. There are several *conjectures* in mathematics, e.g., $P \neq NP$, that attempt to rigorously express this idea but, probably, we do not know yet how to *properly formulate* the problem.

[17] The laws of mechanics are special in this respect – they are built into the motor systems of the brain. *Instinctively*, we always try to find mechanical explanations (models) of everything around us. However, these "brain's laws" are, by necessity, non-Newtonian – true Newtonian mechanics is *instinctively* rejected by most (all?) people even if they formally understand it. (One wonders if this can be tested by experimental psychologists.)

Nägeli, Boltzmann, Guldberg, Waage, van 't Hoff

Karl Wilhelm von Nägeli (1817–1891), a leading botanist of the 19th century, introduced the concept of *meristem* – a group of plant cells capable of division; he also realized the role of sequences of cell divisions in plant morphology. But his fame rests on him being the first on the list of people who failed to understand Mendel's work.

Ludwig Eduard Boltzmann (1844–1906) received his PhD on kinetic theory of gases in 1866 – the year of publication of Mendel's *Versuche über Pflanzen-Hybriden*.

Cato Maximilian Guldberg (mathematician) and *Peter Waage* (chemist) proposed in 1864 the *law of mass action* in chemical kinetics that was logically similar to Mendel's laws; their work went unnoticed until it was rediscovered by van 't Hoff in 1877.

Jacobus Henricus van 't Hoff (1852–1911) was the first winner of the Nobel Prize in Chemistry for his discoveries in chemical kinetics, chemical equilibrium, osmotic pressure, and stereochemistry.

The significance of Mendelian hybridization data, that are similar to the *simple volumes ratios property*, also called *the law of combining volumes* of reacting gases, discovered by Gay-Lussac and reported in his 1808 article

Sur la combinaison des substances gazeuses, les unes avec les autre

would have been apparent to these people. (Gay-Lussac's law says, in Avogadro's words, that

> *les combinaisons des gaz entre eux se font toujours selon des rapports très simples en volume, et que lorsque le résultat de la combinaison est gazeux, son volume est aussi en rapport très simple avec celui de ses composants.*)

Boltzmann especially, a proponent of atomic theory of matter and energy, would have been delighted by Mendel's idea of *atoms of inheritance* conceived with logic similar to that involved in Avogadro's derivation of the Gay-Lussac law from his atomic conjecture:

> *... the number of integral molecules in any gases is always the same for equal volumes, or always proportional to the volumes.*

This is nowadays called the Avogadro Law. It goes along well with the Gay-Lussac law if you assume that chemical substances are composed of distinct mutually identical units (atoms or molecules) but one cannot a priori exclude these units from being *infinitesimally small* in the sense of Leibniz and, accordingly, the *number* of molecules in a finite volume of gas being *infinitely large*. (Such *infinitely large/small* numbers are describable nowadays in the language of *non-standard analysis*. For instance, the number $\mathbf{SToP}$ that we met may be regarded as infinitely large and particles of mass $\mathbf{SToP}^{-1}$ as infinitesimally small.)

But this *number* (defined nowadays as the number of atoms in 12 grams of pure carbon ^{12}C) happened to be finite and not terribly large: the (*Avogadro*)

number N_A of molecules in ≈ 22.4 liters of a gas at 0°C and 1 atm pressure is $N_A \approx 6 \cdot 10^{23}$. This also $\approx$ equals the number of molecules in 18 g of water H_2O of molecular weight ≈ 18. If atoms were one-millionth as large, with $N_A = 6 \cdot 10^{41}$ instead of $6 \cdot 10^{23}$, the idea of atoms could forever remain only an idea.[18] (If atoms were so small, then the age of the Universe – a few billon years – would be too short for the 1 μm cell and, consequently, for organisms of our size, to evolve. Then even the *idea* of atoms could not be possible – no brains no ideas ... unless small atoms would allow brainy organisms of 1 μm in size.)

Avogadro's ideas on atoms had no better luck than those of Mendel on genes – they were generally accepted only years afterwards. But if the problem with Mendel was because his contemporary biologists were not up to his level of science and mathematics, the physicists of the 19th century were skeptical about atoms because of their (physicists') high standards for acceptance of new ideas. Faraday, for example, who understood the problem of atoms as much as Avogadro did, says:

> *it is very easy to talk of atoms, it is very difficult to form a clear idea of their nature, especially when compounded bodies are under consideration.*

(The logical/philosophical problem of atoms had also been very much on the mind of Lavoisier; today, we know that the concept of atoms is self contradictory in the context of classical (non-quantum) physics. But in the natural sciences, unlike how it is in mathematics, a contradiction does not necessary invalidate an idea.)

Atoms and molecules were always on physicists' minds. For example, Maxwell writes:

> *Assuming that the volume of the substance, when reduced to the liquid form, is not much greater than the combined volume of the molecules, we obtain from this proportion the diameter of a molecule. In this way Loschmidt, in* 1865*, made the first estimate of the diameter of a molecule. Independently of him and of each other, Mr. Stoney in* 1868*, and Sir W. Thomson, in* 1870*, published results of a similar kind, those of Thomson being deduced not only in this way, but from considerations derived from the thickness of soap-bubbles, and from the electric properties of metals.*
>
> *According to the table, which I have calculated from Loschmidt's data, the size of the molecules of hydrogen is such that about two million of them in a row would occupy a millimeter, and a million million million million of them would weigh between four and five grammes!*

Maxwell might not have been 100% certain *if there is any truth in the dynamical theory of gases ...* but eventually, atoms won under efforts by Boltzmann, and when Einstein and Smoluchowski wrote down an equation for describing the

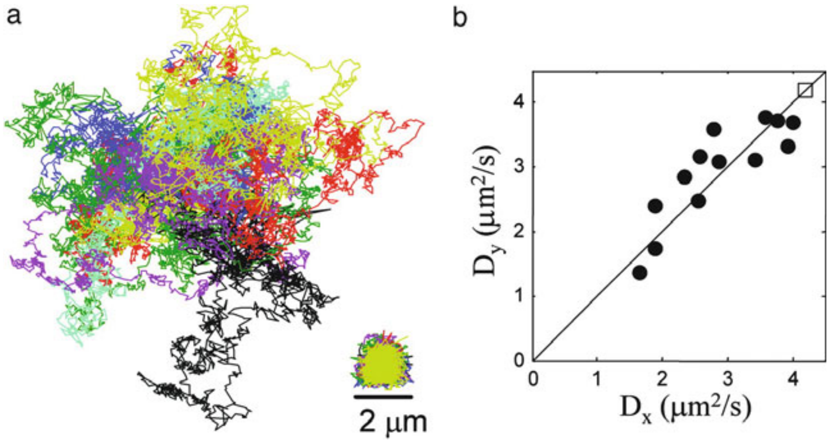
[18] 10^{-6} in linear size makes 10^{-18} in volume and therefore in mass the exponent is $18 = 41 - 23$.

stochastic, called *Brownian*,[19] motion of microscopic particles suspended in a fluid that allowed an evaluation of "the size of atoms".

The idea behind this equation is old and simple:

... small compound bodies ...
are set in perpetual motion
by the impact of invisible blows ....
The movement mounts up from the atoms
and gradually emerges
to the level of our senses.

TITUS LUCRETIUS, 50 BCE (?)



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But it took nearly two thousand years to turn poetry into science by translating these lines to mathematical language.

This "translation" was performed in different contexts independently (?) by: *Thiele* (1880), *Bachelier* (1900), *Einstein* (1905), *Smoluchowski* (1906), *Wiener* (1923), where the mathematics of this translation was not terribly far from *the multiplication table type*.

(The idea would have been apparent, to Pascal and to Buffon, while Euler and/or Laplace, would have had no problem performing the detailed computation, IF either of them had put his mind to it.)

Is it sheer luck or is it a consequence of the *weak anthropic principle* that Brownian displacement of particles is visible with an optical 1000× microscope?

[19]Robert Brown (1773 –1858) did not discover Brownian motion but he gave a detailed description of a cell nucleus and of other cellular structures. Brownian motion was systematically studied for the first (?) time by Jan Ingenhousz around 1785. (Ingenhousz also pinpointed the role of sunlight in photosynthesis.)

Molecules are too small to be detectable by the human eye in the visible light spectrum, but their impacts are felt by objects of intermediate size.

Say, water molecules, $\approx 0.3 \cdot 10^{-3} \mu$m, are *only* (!) 1000 times smaller (twice on the logarithmic scale normalized by our eyesight) than the smallest optically distinguishable objects/displacements, $\approx 0.2\,\mu$m. A $1\,\mu$m bacterium, for example, scaled by the factor 1 000 000 to human size ($1\,\mathrm{m} = 10^6 \mu\mathrm{m}$) would "see" molecules of water as grains of sand.

And water molecules move fast: Their (square) average speed is approximately $650\,\mathrm{m}/s \approx 1000^3 \cdot 0.5\,\mu\mathrm{m}/s$ at the room temperature. (Muzzle velocity of a bullet varies 200–1200 m/s for most guns.)

What one observes in reality are displacements of particles suspended in water under *multiple* collisions with water molecules where disproportionally many of them move in a certain direction. The frequency and average size of these displacements is related to the Avogadro number (or, essentially equivalently, to *Boltzmann constant*) via the Einstein–Smoluchowski formula (the reciprocal to the *Avogadro number* corresponds to "the volume size of an atom.")

Performing a measurement and an accurate evaluation of the Avogadro constant was by no means simple – this was achieved by Jean Baptiste Perrin and his team around 1913–1914 with the resulting value for the Avogadro number: $6.03 \cdot 10^{23}$. (The currently accepted value is $6.0221 \cdots \cdot 10^{23}$ that is obtained with X-ray crystallography of silicon single crystals.)

Do the laws of classical (*non-molecular*) *genetics*, and/or the *classical laws* of physics and physical chemistry of the 19th century still have something new and interesting to offer to us?

A scientist would find it unlikely but a mathematician may think otherwise – we do not feel that anything is understood, unless it is expressed in the language of 21st-century mathematics. But even 20th-century mathematical rendition of something as innocuous as the 18th-century *ideal gas laws*, (e.g., in terms of Riemannian and/or integral geometry) seems non-trivial. (The 20th-century mathematics "dissolves" the law of inertia in the "matrix" of geodesic flows over spaces with affine connections; we expect something more interesting from such a "matrix" in the 21st century.)

Mathematics also serves as a *cord* that ties laws to observations – the *empirical truth* of a law is not seen in passively observed *facts*. Mathematical ideas enter, often implicitly, into design and interpretation of experiments.

Nobody, for example, ever saw a *body* moving by itself with constant velocity along a straight line – the law of inertia stands in stark contradiction with what you observe. Yet, you cannot draw a mathematically consistent/elegant picture of mechanics without this law; e.g., you cannot write down an esthetically acceptable (say, given by analytic functions) formula compatible with Galileo's rolling balls on inclined planes without *taking* the balls velocity being *constant* for the inclination angle *zero*.

5. Truth

La pensée ne doit jamais se soumettre,
ni à un dogme, ni à un parti, ni à une passion,
ni à un intérêt, ni à une idée préconçue.

Henri Poincaré

When you point out something naive or just plain silly in writings by a scientist of the past, one may say that you yourself are being foolish by judging an idea of yesterday from the standpoint of today.

Well ... this would be convincing if not for people like Lavoisier, Claude Bernard, Faraday, Poincaré. They had fine ears for the ringing of scientific truth – they never (almost never?) said anything senseless; mistaken – maybe, but never trivial, vacuous, or intrinsically inconsistent.

Qu'un objet change d'état ou seulement de position,
cela se traduit toujours pour nous de la même manière
par une modification dans un ensemble d'impressions.

Henri Poincaré, La science et l'hypothèse, 1902

Everything shined at the touch of Poincaré's mind, be it mathematics, physics, or anything else. His analysis of the space perception in Chapter IV of *La science et l'hypothèse* remains unrivaled in its depth and clarity. His conjecture on how the brain reconstructs the *a posteriori Euclidean* geometry of external space with its rotational (orthogonal[20]) symmetry from the input of moving retinal images, that amazingly agrees with discoveries of 20th-century neurophysiology, stands as a challenge for experimental (mathematical?) psychologists of the 21st (22nd?) century.

... science d'observation est une science passive.
Dans les sciences d'expérimentation, l'homme ...
provoque à son profit l'apparition de phénomènes,
qui ... se passent toujours suivant les lois naturelles,
mais dans des conditions que la nature
n'avait souvent pas encore réalisées.

... l'astronome fait des observations actives,
c'est-à-dire des observations provoquées par
une idée préconçue sur la cause de la perturbation.

Claude Bernard, Introduction à l'étude
de la médecine expérimentale, 1865

[20] The *orthogonal group* $O(3)$ of rotations of 3-space is the basic *non-commutative atom* of our world geometry. Mathematicians' penetration into its amazing structure took two-and-a-half millenia – from *Pythagoras' theorem* to the theory of *Lie groups* developed at the turn of the 20th century and the present day Euclidean (elliptic) gauge theory over 4-D spaces. Amazingly, the brains of all actively moving animals on Earth have developed (by learning?) fair models of this structure.

If you quote a writing by Claude Bernard in an audience of mathematicians they will assume this must be Poincaré, physicists would think of Einstein or Feynman, and biologists of Darwin. But some may recall having seen these very words in a memorandum signed by several Nobel laureates in the last issue of Nature.

One hardly can add anything of substance to what Claude Bernard had already said about experimental science and what he says applies to theories as well, since a design of a logical argument is similar to that of an experiment.

Claude Bernard insisted that an *active science* depends on experiments and these must *control rather than confirm* our ideas. He could have continued by saying that an *active logic* must similarly control rather than confirm our ideas. We should not decide beforehand where the argument will bring us; yet, the rules of our logic must be set in advance. Then we go through an argument step by step and accept whatever comes out of it even if this contradicts our original ideas.

It is almost impossible to design such arguments without mathematics, where *going through an argument* being furnished by something like a proof of a theorem or "just" a computation; it is amazing how often the result of a simple computation may contradict your intuition.

An instance of that is the (Mendel)–Hardy–Weinberg formula we met before that expresses the (counterintuitive) idea that (naively understood) *evolution by natural selection* may stabilize on the second round of reproduction.[21]

Let us give another example that simultaneously illustrates Claude Bernard's remark that *averages confuse, while aiming to unify*.

Suppose that the average success of a certain brain surgery performed in two hospitals H_1 and H_2 depends on whether the father of a patient was right or left handed.

The *right* patients are sent to H_1, since

$\boldsymbol{H_1}$ has **better average success** for ***right***.

The *left* patients are also sent to H_1, since

$\boldsymbol{H_1}$ has **better average success** for ***left*** as well.

But if you do not know whether you are in the *left* or *right* category you are sent to H_2, since

$\boldsymbol{H_2}$ has **better average success** for ***right*** + ***left*** together.

Perplexed?[22] Then look at the following formula for the average success rate S_{r+l} for *right* + *left* together in terms of those for *right* and *left* separately, denoted

[21] The mathematics of this formula necessitates the passage from Darwinian idea of *variation* to the more abstract (and more adequate) concept of *mutation* that was envisioned by Maupertuis in his picture of Life. (The term *mutation* for suddenly appearing variations was introduced by Hugo de Vries in his two-volume *The Mutation Theory*, 1900–1903.)

[22] All mathematicians I know refused to believe this could ever happen, but those of my friends who were not certain about their math pretended they could admit such possibility.

S_r, and S_l, where N_r and N_l are the numbers of the *right* and *left* patients who have undergone this surgery,

$$S_{r+l} = \frac{N_r S_r + N_l S_l}{N_r + N_l}.$$

Now you see how this *can* happen in certain cases: S_{r+l} depends not only on S_r and S_l but also on N_r and N_l, where these numbers may be quite different in the two hospitals.

Rephrasing Einstein – *the laws of mathematics are certain even when they refer to reality; it is which reality they refer to that is not certain.*

... à ne chercher la vérité que dans l'enchaînement naturel des expériences et des observations, de la même manière que les mathématiciens parviennent la solution d'un problème ...

Antoine Lavoisier, Discours préliminaire au Traité élémentaire de chimie, 1787

Je vous dénonce le coryphée des charlatans, le sieur Lavoisier.

Jean-Paul Marat, l'Ami du Peuple, janvier 1791

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What unworthy motives ruled for the moment under high sounding phrases.

Michael Faraday

Lavoisier was arrested in November 1793, accused of selling watered-down tobacco and guillotined on 8 May 1794 at the age of 50.

Lagrange[23] commented on Lavoisier's execution:

> *Cela leur a pris seulement un instant pour lui couper la tête, mais la France pourrait ne pas en produire une autre pareille en un siècle.*

Claude Bernard was born in 1813 and Henry Poincaré in 1854.

But who was Jean-Paul Marat?. Marat performed numerous experiments and published work on fire, heat, electricity, and light. This was not taken seriously by the contemporary French "academic establishment", in particular, by Lavoisier and Condorcet.

Some historians sympathetic to Marat argue that, according to certain evidence, he was a knowledgeable and dedicated scientist who did not get a fair hearing at the French Royal Academy and that a comparison of Marat to somebody like Mesmer is unfair. (In 1784, Lavoisier and Benjamin Franklin, who were heading *la Commission royale d'enquête sur le magnétisme animal*, conducted the historically first controlled clinical trial and disproved Mesmer's claims.)

This might be true, but, apparently, Marat, however *knowledgeable*, was not appreciative of the extent of his *non-understanding*. Thus, he published a volume of a few hundred pages on his study of electricity. Lavoisier, on the other hand, who believed that

> *l'électricité ne jette pas seulement de la lumière sur les effets du tonnerre, elle sert encore à expliquer un grand nombre des opérations de la nature,*

and who had been pondering on electricity most of his life, had published virtually nothing on it. Lavoisier thought at some point that

> *le fluide éctrique et le fluide magnétique sont eux-mêmes composés, comme l'eau, de deux fluides plus subtils encore, qui sont susceptibles de se réunir et de se séparer, et, en quelque façon, de se neutraliser l'un par l'autre,*

but he could not find sufficient experimental evidence for his ideas and remained uncertain on the nature of electricity; but Marat believed he understood what electricity was. One tends to agree with Lavoisier's assessment of Marat's work as being insignificant.[24]

Marat refused to accept Lavoisier's judgement and hated Lavoisier, but he was not, however, *directly* involved in Lavoisier's condemnation: Marat was assassinated in July 1793.

---

[23]Lagrange (1736–1813), born Giuseppe Lodovico (Luigi) Lagrangia, was a mathematician. The imprints of his analytic rendition of the *least action principle* (more precise than that by Maupertuis and more general than that by Euler) are pronouncedly visible as *Lagrangians* (instead of Newtonian forces) *of dynamical systems* in many segments of mathematical physics along with their younger sisters *Hamiltonians*.

[24]Lavoisier could have said what he thought of Marat in Pauli's words: *I don't mind your thinking slowly; I mind your publishing faster than you think.*

6. Life

*... strutture di ossa per uomini, cavalli o altri animali, che
potessero sussistere e far proporzionatamente gli uffizii loro,
mentre tali animali si dovesser agumentare ad altezze immense .*

Galileo Galilei, Discorsi e Dimostrazioni
Matematiche Intorno à Due Nuove Scienze, 1638

The science of *allometry*, the birth of which you witness reading the above lines, still mystifies us by some of its findings. Why, for example, does the metabolic rate R of an animal of mass M closely follow

Kleber's 3/4*-law:* $R \sim M^{3/4}$*?*

*La Nature contient le fonds de toutes ces variétés,
mais le hasard ou l'art les mettent en oeuvre.*

Pierre-Louis Maupertuis, Vénus physique, 1745

Maupertuis pinpointed the role of natural selection in evolution, outlined a (not quite Mendelian) picture of heredity, sketched the theory of mutations and suggested a germ plasm mechanism close to that of Weismann. Since he was a mathematician, biologists are justified in having ignored his ideas.

... *que de machines ... sont renfermés dans cette petite partie de matière
qui compose le corps d'un animal! ... combinaisons ... de principes,
que nous ne connaissons que par des résultats si difficiles à comprendre,
qu'ils n'ont cessé d'être des merveilles que par l'habitude que nous avons
prise de n'y point réfléchir!* [25]

Georges-Louis Buffon, Histoire générale des animaux, 1749

Buffon applied integral calculus to measure chances of geometric random events. For example, if one throws a needle of unit length to the plane divided into parallel strips of unit width, then the probability that the needle will cross a line between two strips equals $2/\pi$ for $\pi = 3.14\ldots$ by Buffon's formula.

(This brought to life the fields of integral geometry and of geometric probability; also this inspired the currently accepted mathematical foundation of probability theory laid out by Kolmogorov[26] in 1933.)

Buffon designed efficient lenses for lighthouses.

Buffon suggested a construction of concave mirrors that has been in use for two centuries afterwards.

---

[25] Was Buffon able to see this because of his mathematical background? Was he the first to realize it? Was it possible at all to appreciate Life's complexity and our powerlessness in structural representation of live entities prior to the development of molecular biology in the last few decades?

[26] Beside modern probability, several other branches of 20th-century mathematics and mathematical physics grew out of work by Kolmogorov including *KAM theory* in classical (Hamiltonian) mechanics, a stochastic theory of *turbulence*, and *entropy* theory of dynamical systems.

Buffon offered the first scientific scenario of formation of planets, namely from a collision of a comet with the sun.

Buffon evaluated the age of Earth at several (hundred?) million years on the basis of sedimentation rates and argued that at least $\approx 75\,000$ years were needed for the Earth to cool down from the molten state, as he calculated by "rescaling" the results of heating and cooling iron balls.

(The idea of hot molten Earth was present in the writings of Descartes and Leibnitz.)

Buffon gave a definition of species:

> *One should consider as being of the same species that which by means of copulation perpetuates itself and preserves the similarity of that species … If the product of such mating is sterile, as is the mule, the parents are of different species. Any other criterion, particularly resemblance, is insufficient … because the mule resembles the horse more than the water spaniel resembles the greyhound.*

This is rather different from how it was formulated by the natural philosopher John Ray who defined species by their

> … *distinguishing features that perpetuate themselves in propagation from seed.* (1686)

Apparently, Buffon, a mathematician in the depth of his heart, must have been mystified and fascinated by the remarkable fact that the obstruction to interbreeding shows in the *second* generation, defined an *equivalence relation* between groups of organisms that allowed an introduction of the concept of species.[27]

Buffon would be happy to see that his kind of mathematically active scientific thinking returned to taxonomy in biology at the end of the 20th century with the comparative study of genome sequences.

Buffon applied his definition to prove the unity of humanity:

> *The Asian, European, and Negro all reproduce with equal ease with the American.*

Buffon writes:

> *… one could equally say that man and ape have had a common origin like the horse and donkey that each family among the animals and vegetables have had but a single stem, and that all animals have emerged from but a single animal which, through the succession of time, has produced by improvement and degeneration all the races of animals.*

[27]In 1942, an influential evolutionary thinker Ernst Mayr reiterated Buffon's definition as

> *… interbreeding natural populations, which are reproductively isolated…*

What would Buffon make of the 20th-century "evolutionary thinkers" who were unaware of mathematics of *clusterization* and who authoritatively stated that having a progeny needs mutual proximity of participating animals? (Never mind plants.)

Buffon, who had already had enough trouble with ecclesiastical authorities, rejected this idea in writing, since it contradicted the biblical version of creation.

Buffon would find it ironic that certain post-Darwinian evolutionary thinkers who have freely borrowed from his ideas, have, nevertheless, inherited a pre-Cartesian respect for *authority* and submission to *ideology*[28] from his ecclesiastical opponents. These thinkers have not admitted Buffon to membership in their club as a *non-believer* in Darwinism.

Buffon overhauled much of the knowledge of his time and developed a view on Nature and Life from a broad scientific perspective. But he wrote down only 36 volumes of his *Histoire naturelle, générale et particulière* out of planned 50 before he died in 1788.

(Buffon, like Einstein, complained that his natural laziness had precluded him from achieving more in science.)

Buffon's idea of evaluating the age of Earth was taken up by William (Kelvin) Thomson in 1862 who calculated the rate of thermal diffusion from the Earth's crust of its initially molten state and, thus, estimated Earth being at most of order 100 million years old. (This was in contradiction with the sedimentary geological estimates by Maillet, Lyell, and Darwin.)

Thomson is famous for the concept of *absolute zero*, $\mathbf{0}^\circ\,\mathrm{K} \approx -273.15^\circ\mathrm{C}$ and for the idea of the *heat death of the Universe*:

> *. . . a state of universal rest and death, if the universe were finite and left to obey existing laws. . . . science points rather to an endless progress, . . . involving the transformation of potential energy into palpable motion and hence into heat . . . .*

As an engineer Thomson had greatly contributed to the transatlantic telegraph project.

But Thomson is also renown for his:

> *heavier-than-air flying machines are impossible*

and the characteristically late Victorian:

> *There is nothing new to be discovered in physics now.*

This contrasts with

> . . . *we have no right to think thus of the unsearchable riches of creation . . . .*

by Maxwell (who is often misquoted at this point).

---

[28]We value in ideas their *beauty* and their *novelty*; we appreciate in people the depth and originality of their *understanding* of non-trivial ideas. Carriers of an ideology, on the other hand (often recognizable by *ist* attached to their tails), may be tolerant to those who are confused by what they (the carriers) proclaim – these are potential converts, but they are indignant against anyone who *clearly sees and rejects* their views. Faraday's

> *Nothing* [is] *quite as frightening as someone who knows they are right*

is about these people.

Buffon's direct impact on physics was, probably, rather limited but the flow of ideas emanating from his "Natural History" had been shaping the minds of generations of biologists and natural philosophers up to the middle of the 20th century.

> *It is in relation to the statistical point of view that the structure of the vital parts of living organisms differs so entirely from that of any piece of matter that we physicists and chemists ...*
>
> *... living matter ... is likely to involve "other laws of physics" hitherto unknown, which however, once they have been revealed, will form just as integral a part of science ....*
>
> Erwin Schrödinger, What Is Life? 1944

> *The principle of continuity renders it probable that the principle of Life will hereafter be shown to be a part, or consequence of some general law.*
>
> Charles Darwin, Letter to George Wallich, 1882

Whereas Darwin dreamed of some *mathematical/philosophical* Principle of Life, Schrödinger wanted to make sense of Life in the light of Physical Laws.

But Life breaks physical symmetries/regularities and creates new ones[29] of quite different nature, e.g.,

the striking regularity with which the same hybrid forms reappear.

The laws of physics, obviously, restrict possible structures/behaviours of living systems as much as the rules of chess restrict the moves of pieces on a chess board.

But moves of a master in a chess match are no more accountable for by the rules of chess than mathematical theorems are representable by reflections of the laws of logic.

These rules, one might argue, well account for the statistics of the moves of somebody who plays chess for the first time in his/her life but not of a player with a couple of years of experience. This is how it is with most *physical systems* – they start their games essentially from level zero, where the laws of physics apply.[30]

But the game of life on Earth has been played by Nature for a few billion years. The laws of physics, as Galileo pointed out, tell you that there are no flying elephants but they cannot help you to draw the path that brought *walking* elephants to Earth. Compatibility with the laws of physics is a minuscular part of the reason for existence in biology.

---

[29] It is not always clear what should be called "breaking" and what is "creating" a symmetry. Does, for example, selection of a particular *chirality* by biological systems break or create stochastic symmetry in ensembles of molecules?

[30] The second law of thermodynamics is an exception: it applies, as Boltzmann pointed out, only to *prepared states* of physical systems, prepared by their *history*. This law appears to *Naive Mathematician* as a physical corollary to a general mathematical theorem, a theorem nobody has managed to formulate yet.

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It is not accidental that when Buffon and Schrödinger try to speak of Life in the language of physics they start speaking poetically.[31] The great painting of Life is not about the physics of the canvas and the chemistry of dyes.

But don't physical and biological theories share the hallmark feature of "true science": *falsifiability*?

> *You don't need* 100 *famous intellectuals to disprove a theory. All you need is one simple fact.*
>
> Albert Einstein

For instance, think of *Galileo's law of falling bodies*:

if you let go down some objects ✤, ✺ *and* ❀ *you hold in your hand, say of* 300, 30 *or* 3 *grams of weights, they all reach the floor in* $\approx \frac{1}{4}$ sec *essentially independently of their weights.*

Would the edifice of classical mechanics collapse if, in some experiment on the sea level of planet Earth, one of these objects, for instance ✺ of 30 grams of weight, had remained in the air for 10 seconds before reaching the ground?

The answer could be "*Yes, it would*" IF Earth were bare of life. And to get the correct answer, no 100 intellectuals are needed, a single *healthy adult sparrow* for ✺ will do.

But there is no way to say what "(bare of) *life*" signifies in the language of physics.

The main impediment to "*reducing Life to Physics*" is not that we lack some "Law of Nature", but incompatibility of the two contexts underlying the languages appropriate for description of physical and of biological phenomena. What we call *physics*, or rather *theoretical physics*, is not only a collection of mathematical models, but also a set of *tacit* rules of when, where, and how these models should be related with results of experiments.

[31] This is reminiscent of Trevize in Asimov's *Foundation and Earth* who sees the Milky Way Galaxy as an unfolding of Cosmic Life.

One can imagine a universe with physical laws very different from ours; yet, inhabited by somebody very much like ourselves, e.g., a virtual world of a sophisticated computer program. Probably, an outline of the ways of Life can be seen in how *mathematical order* (*mysteriously*) "assembles" from simple "logical pieces", not in the pieces themselves.

But is there an abstract notion of *system* suitable for biology?

Can a cell be regarded as a *single* physical system?

What is a true *Grothendieck-style*[32] definition of "*single unit*" in biology?

A few facets of life can be described in the language of probabilities that is commonly used in physics, but these descriptions are dissimilar to the pictures that we see in physics and in chemistry as Schrödinger says. *Statistics* of LIFE, unlike *statistics* of *non-Life*, is characterized by

- *Improbably* ***high multiplicity*** *of occurrence of supposedly rare seemingly independent events*,

  such as the presence of thousands of *nearly identical* randomly looking *complex* protein molecules in a cell. (But hardly *two complex* snowflakes found in Antarctica are ever completely alike.)

  On a different scale, and for a somewhat different reason(s), there may be trillions of *copies* of *thousands residues long* polynucleotide molecules (DNA and RNA) and/or of viral particles in a pond of water, not to mention seven billion enormous multi-molecular aggregates of almost indistinguishable compositions and shapes, that bipedally locomote themselves on the surface of Earth.

This *improbable multiplicity* phenomenon is partly due to

- *Amplification of rare and/or low energy events*, e.g., of advantageous mutations in populations and of chemical signals in cells. (The latter would look similar to a chain reaction of combustion ignited by an accidental spark, if not for *key/lock mechanisms* ever-present in biological systems.)

This amplification is also responsible for

- *Gross discrepancy between "statistically averaged", "significant", and "typical" events/behaviours.*

  Processes in Life often run as lotteries – no one's gain is close to the average – in stark contrast with comparably large *stochastically homogeneous* physical systems, such as the bulks of gases and liquids where *typical* tends to be close to *average*.

In many (but not all) cases the above features are associated to *information* that is *encoded* in a biological system and that can be *transmitted* from one biological (sub)system to another.

---

[32] Alexander Grothendieck (1928–2014) is the originator and crystallizer of a body of fundamental concepts of 20th-century mathematics.

In fact, "information" is a commodity more valuable for Life than energy: The transfer of *sequential information* in the course of replication

$$\mathrm{DNA} \to 2 \times \mathrm{DNA}$$

is a defining factor of Life on Earth.

And the "*circulation of information flow*(*s*)" that makes the cell tick starts with the sequential transfers

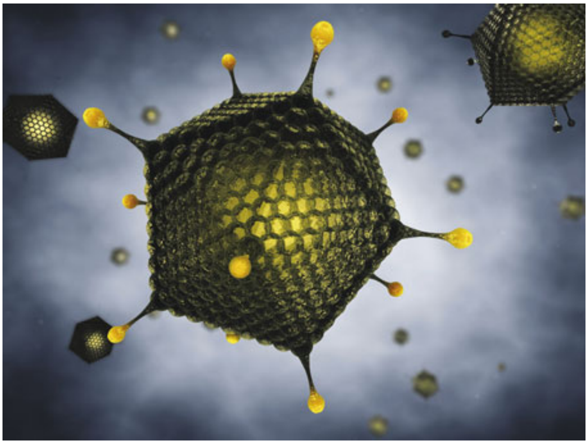
$$\mathrm{DNA} \to \mathrm{RNA} \to \textit{polypeptides}.$$

These transfers are "formal" and (essentially) expressible in terms of Shannon's style of information theory. But there would be no Life if not for *macromolecular folding*. In a way, Life is created by

$$\textit{polypeptides} \overset{\text{folding}}{\rightsquigarrow} \textit{proteins}.$$

Albeit folding is an essentially physical process,[33] physics per se cannot tell how and why it works, because, besides lacking precise "physical/chemical formulae" for amino acid residues' interactions in the aqueous environment,[34] folding applies only to those *rare* polypeptide chains that were "selected by Life" for their specific roles in the cell.

Neither physics nor present day mathematics can identify and formally describe polypeptide sequences that would fold to "*potentially live-cell proteins*".



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Even worse conceptually, we possess no mathematical framework where we could formulate the idea of "*creative reduction of information by folding*".[35]

[33] Folding of some proteins needs "biological assistance" in a cell.

[34] There is not even a satisfactory model of water itself on the nanoscale.

[35] Apparently, the bulk of the message carried by the amino acid sequence tells a protein P how it should fold rather than what it is supposed to do upon folding. This "folding information"

Similarly, there is no available mathematical formalism for the expression of the "*economy of information principle*" that dictates the icosahedral symmetry of viral particles (such as in the picture blow), where this information is what is needed to genetically encode an assembly of such particles in the rotationally symmetric physical space.

And evolving the ability of termites to build skycrapers was, probably, grossly facilitated by the possibility of a *symmetric* implementation of the building program in *identical* genomes of the builders.

Is this kind of *information* a precursor of a non-trivial mathematical concept or it will forever remain a poetic metaphor?

Physics has no means for addressing these kinds of questions. But why – one asks with an insinuating smile – were there fundamental contributions to biology by physicists and not so many, if at all, by mathematicians?

I could give you an answer, but this would be ... opinion.

7. Evolution

Nothing in Biology Makes Sense Except in the Light of Evolution.[36]

This is the title of a 1973 article by Theodosius Dobzhansky.

We now may be able to understand it [evolution] *in biology.*

Jacques Monod, On the Molecular Theory of Evolution, 1975

We have an excellent idea of the core genetic makeup
of the last common ancestor of all bacteria
that probably lived more than 3.5 billion years ago.

Eugene Koonin, The Logic of Chance, 2011

The latter is not another excellently contrived just so story but an outcome of statistical analysis (of *multiple alignments*) of genomes from data bases that have been accumulated at a *petabyte* (10^{15} bytes) rate for the last ten years. From the petabyte hight, Dobzhansky's *Light* and Monod's *understand* appear as dim as the early 20th-century picture of evolution, often called *modern evolutionary synthesis*, gauged by the molecular standards of the 1970s.

And the ideas of the 18th and 19th centuries serve for molecular biologists only as references to the names behind them. Yet, such concepts as *reproduction – inheritance – selection – competition* seem to carry still undeciphered messages that we would be able to read if we could translate *natural poetry* of olden days to our mathematical language, preferably, not exactly of *multiplication table type*.

is, for the most part, invisible on the exposed surface of a folded P but the sequence can be recovered when P is unfolded. Thus, the term "reduction" is not quite appropriate.

[36]In mathematical terms, life on Earth becomes *connected*; thus, a topologically *non-trivially* structured entity *only* in the presence of the *time* coordinate. Is this connectivity or the *full* topology (geometry?) of Life on Earth what Dobzhansky calls *sense*?

First forms minute, unseen by spheric glass,
Move on the mud, or pierce the watery mass;
These, as successive generations bloom,
New powers acquire and larger limbs assume;
Whence countless groups of vegetation spring,
And breathing realms of fin and feet and wing.

Erasmus Darwin, The Temple of Nature, 1802

... difficult to believe ... that the more complex organs and instincts should have been perfected, by the accumulation of innumerable slight variations. Nevertheless, ... all organs and instincts are, in ever so slight a degree, variable ... a struggle for existence ...

Charles Darwin, The Origin of Species, 1859

Some pages later Darwin writes:

... it is as easy to believe in the creation of a hundred million beings as of one; but Maupertuis' philosophical axiom "of least action" leads the mind more willingly to admit the smaller number.... Analogy would lead me one step further, namely to the belief that all animals and plants have descended from one prototype.

This sounds almost like a mathematical proposition! Pythagoras would have loved such a quintessential formulation of the idea of evolution. He might have learned this idea from Anaximander and he would have stated it almost as Darwin put it:

Theorem 1. *Every two organisms on Earth, be they plants, animals or humans, have a common ancestor.*

But only Pythagoras' corollary reached us across the chasm of time:

As long as man continues to be the ruthless destroyer of lower living beings he will never know health or peace. For as long as men massacre animals, they will kill each other.

Darwin, who hated slavery of any kind, adds:

Animals, whom we have made our slaves, we do not like to consider our equal.

Lamarck, Darwin, and Wallace had scrutinised and systematised vast amount of material, in part collected by themselves, that led them to believe in transmutation of species and evolution in general. Darwin and his followers chose to present their ideas to the general public in the language of political economy: *competition for resources, struggle for existence*, etc. That was well taken by their Victorian audience acquainted with *The Wealth of Nations* by Adam Smith and prepared for evolutionary ideas by *Vestiges of the Natural History of Creation* published anonymously in 1844 by Robert Chambers.

But **Theorem 1** was truly *proved* only a century later with the advent of molecular biology and sequencing techniques that uncovered *non-ambiguous* structural similarity between molecular architectures of all living cells.

Everyone who is able to *understand* Pythagoras' theorem will also *understand* that such a degree of similarity could be neither accidental nor come from any kind of *convergent evolution*. You do not have to *convince* anybody anymore and can discard *struggle for existence, fittest survives*, and *creative power of selection*. Granted the accumulated sequential and molecular structure data, all you need is *a little mathematics of the multiplication table type* in Hardy's terms.

(The Darwinian *metaphor* of *struggle for existence* applies, with no possibility of being taken literally, to *numbers* $R_1 > 1$ and $R_2 > 1$: The greatest of the two numbers survives in the formula $\sqrt[T]{R_1^T + R_2^T}$ for time T being large. But when struggle for existence is applied to animals the reproduction rates of which are represented by these numbers, it may give you wrong ideas.)

Yet, there are two problems with **Theorem 1**.

1. It logically (and obviously) implies that there was a first *proto-cell* from which all other cellular organisms (viruses?) descended, where *proto* is a shorthand for *I have no idea what the Hell I am talking about*.[37]
2. The second question reads: Where is the hidden beauty? What are the physical/chemical/biological mechanisms besides heredity that drive evolution?

The idea (opinion?[38]) of Natural Selection can be condensed to a single word:

NONE[39]

The *potentially exponential* rate of growth of populations allows *random* variations to cover *all* possibilities; all Nature/enviroment has to do is to *select* those she favors.

Darwin also insisted that consecutive steps of evolution are implemented by *small* variations.[40] These, he argued, are more likely to be non-deleterious; besides, this increases the chance of a (more or less) simultaneous occurrence of several, say two, variations that are advantageous only if they come together.

[37] The Earth is about 4.5 billion years old and *great oxygenation* of Earth's atmosphere happened $\approx$ 2.5 billon years ago. There are $\approx$ 2 billion year-old fossils of multicellular organisms and traces of pre-oxygenic bacteria $\approx$ 3.5 billion years old. But when and what was the age of proto-life?

[38] An assertive expression of an idea that has been around for a few decades classifies as an *opinion*.

[39] This answer brings to mind the famous apocryphal response of Laplace to Napoleon:

Sire, je n'ai pas eu besoin de cette hypothèse.

[40] The Western idea of gradualism can be traced back to Milo of Croton, an associate of Pythagoras, who, around 540 BCE, gradually developed his strength by having carried a calf growing to bull on his shoulders.

(Major transitions in evolution are, contrary to what Darwin would think, associated not with the accumulation of *point mutations* but with *abrupt* and significant genome rearrangements, including *duplications of genes and of whole genomes*. For instance, the divergence of the lineages of humans and chimpanzees, that occurred about 5 million years ago, was, most likely, triggered by the *fusions of two chromosomes*: great apes have 24 pairs of chromosomes while humans have only 23 pairs.)

Besides, Darwin suggested several scenarios of how particular patterns of evolution, e.g., that of the vertebrates' eyes, could have been gradually achieved by selection.

But Pythagoras would point out that *X is not impossible* does not imply *X is true*, e.g., for *X* being *selection and nothing else*,[41] and that the true biological problem is concealed by dismissive *besides* attached to *heredity*.

In fact, until the development of cellular/molecular biology that started at the end of 19th century, nobody was close to imagining the immensity and the beauty of the hidden structural complexity of Life; hence, of heredity. Even today, some may argue, we have not reached the point where *I know that I know nothing* coming from a biologist will be not just words. But as recently as at the beginning of the 19th century, some, e.g., Lamarck (but not Maupertuis) earlier, believed that spontaneous generation of worms from dirt was plausible.

There was a logical reason for this belief: A presence of rich flora of parasitic worms in the intestine of Adam would contradict the idea of Eden. But it became clear to (almost) everybody by the end of the 19th century that "spontaneous generation" of a single cell of a worm is no more probable, than spontaneous emergence of *great pyramid of Giza* – the first of the seven wonders of the world, out of stones, sand, and dirt by some natural physical process some forty-five centuries ago.

Eventually, accumulation of data on *cell division*, in particular, the discovery of *meiosis* (cell division for production of *gametes*, e.g., sperm and egg cells in animals) by Oscar Hertwig in 1876, led to the *germ plasm* idea proposed by August Weismann around 1890 that turned evolutionary theory toward *active science* in the sense of Claude Bernard.

In modern terms, the *Weismann principle* says:

genomes vary but organisms are selected.

(To confirm his idea and to disprove Lamarck, Weismann was chopping off the tails of several hundred mice for ≈ 20 generations and recorded no mouse born without a tail.[42])

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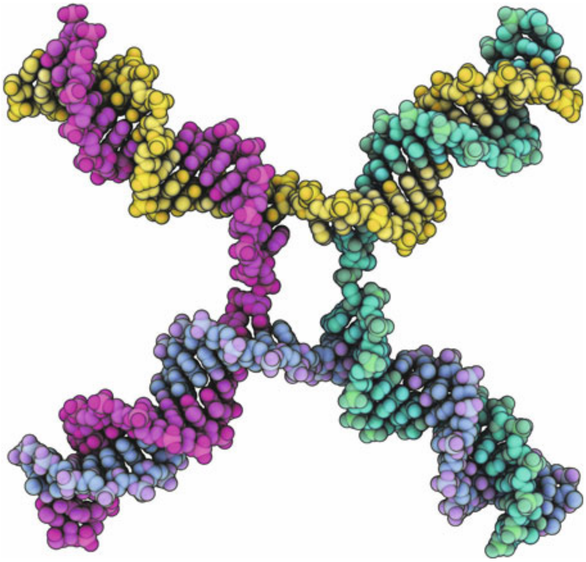
[41] Darwin himself wrote in the 1872 edition of *The Origin of Species* that he was

... convinced that natural selection has been the main but not the exclusive means of modification.

This, however, like prophecies by the oracle of Delphi, is open to many interpretations.

[42] Lamarck would point out that if Weismann had done it to a hundred million mice for a hundred million generations, then the tails would degenerate as do the eyes of fish living in dark caves. He

Soon thereafter, Mendel's ideas was rediscovered and an explosion of genetics followed by that of molecular biology began. The focus of the evolution problem shifted from organisms to genomes.



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We want to learn how genomes "fly", how their "engines" work. This, we believe, will bring us closer to understanding how they fare in the adverse winds of competition and selection. (But doesn't this metaphor go too far when applied to natural selection? Granted, for instance, that the main factor for having a safe flight is a *selection* of an air company, can you seriously say that this selection is the main factor that keeps an airplane safely in the air? If not, how can you take natural selection as the *main* factor in evolution as Darwin says?)

A rough quantitative picture of genome evolution can be seen with simple mathematics of *biased* (non-symmetric) *random walks* on *finite* (but very large) *graphs* with *absorption*, where the vertices of the graphs represent genomes and absorption corresponds to extinction/selection. But we are still far from *formulating* **Theorem 2**.

An amazing thing about old fashioned naturalists, especially Darwin, is that with *no support by an experiment* and/or with *no quantitative reasoning*,[43] they

would also conjecture that if Weismann had done it to a hundred million generations of lizards then, instead of regenerating their tails, they would start growing second heads with one not being enough to comprehend the scientific significance of such experiment.

[43] Because of these shortcomings, the Darwinian scheme of evolution may appear exceedingly naive. For instance, biochemist Ernst Chain (1906–979), one of the main contributors to isolating and applying penicillin, writes: *... variants do arise in nature and ... their emergence can and*

could built consistent pictures of large fragments of Nature and sometimes could do it (literarily) a hundred times better than the most luminous physicists, astronomers, and mathematicians of their time.

For example, Darwin and his friend, geologist Charles Lyell, following ideas originated in the work by James Hutton, estimated (somewhat differently) the age of Earth to be at least several hundred million years. (About 150 years earlier Benoît de Maillet evaluated the age of Earth as two billion years by estimating the rates of sedimentation and formation of Earth's crust, but his argument is judged as having been poorly founded.)

On the other hand, William Thomson (Kelvin), Hermann Helmholtz, and Simon Newcomb came up with about thirty million years by evaluating the time needed for Earth to cool from the molten state and the Sun to be heated and sustained by gravitational contraction. If physicists had taken Hutton, Lyell, and Darwin seriously, they might have arrived at the matter-energy idea several decades before Becquerel's 1896 discovery of radioactivity and Einstein's $E = mc^2$ of 1905.

And pondering over

> *It is not the strongest of the species that survives, nor the most intelligent that survives. It is the one that is the most adaptable to change.*

one starts wondering if these words may come true in a few hundred years from today: The Earth will be taken over by the most adaptable ones – protozoa, bacteria, and viruses after Man will have done with multicellular life.

(The reader may be relieved: The above is a common *misquotation* of Darwin; he was neither that stupid nor *that* gloomy on the future of mankind.)

About Hutton, Lyell, Helmholtz and Newcomb

James Hutton (1726–1797) recognized the role of subterranean heat in the creation of new rock followed by the gradual process of weathering and erosion on a very long geological time scale – *la route éternelle du temps* as Buffon says in *les Époques de la Nature*, published in 1778. (Many of these ideas, including evolution, appear in *Protogaea*, written by Gottfried Leibniz between 1691 and 1693, and published in 1749. For example, Leibniz writes: *The globe of the earth ... has hardened from liquid, light or fire being the motive cause.*)

Hutton thought of Earth geo-dynamics as being cyclical, with time stretched without limit into past and future. Concerning evolution of life, he accepted the dominant role of selection in what we now call *microevolution* but not, as Darwin and Wallace did, in *macroevolution.*

Charles Lyell (1797–1875) was a proponent of gradualism in geology whose uniformitarian ideas influenced Darwin.

does make some limited contribution towards the evolution of species. The open question is the quantitative extent and significance of this contribution.

Hermann von Helmholtz (1821–1894) invented the *ophthalmoscope* for examining the inside of the eye and the *Helmholtz resonator* to identify frequencies of sound waves.

Helmholtz measured the speed of propagation of nerve impulses and he developed mathematical and empirical theories on depth, color, sound, and motion perceptions.

Helmholtz formulated the *law of conservation of energy* in his mechanical foundation of thermodynamics, where he also introduced *Helmholtz free energy*.

Being a rare scientist whose discoveries had found immediate uses, he nevertheless states:

> *Whoever in the pursuit of science, seeks after immediate practical utility may rest assured that he seeks in vain.*

In Maxwell's words:

> [Helmholtz] *... Who prosecutes physics and physiology, and acquires therein not only skill in developing any desideratum, but wisdom to know what are the desiderata.*

Simon Newcomb (1835–1909) conducted a precise measurement of the speed of light. He discovered what is now known as *Benford's law*: More numbers, taken from "real life" data, will begin with 1 than with any other digit. Newcomb believed that the astronomy of his time was nearing the limit and, like Thomson, he was skeptical about *flying machines*.

> *... those which depart most from the best adapted constitution,*
> *will be the most liable to perish, while, ..., those organized bodies,*
> *which most approach to the best constitution for the present*
> *circumstances, will be ... multiplying the individuals of their race.*
>
> James Hutton, An investigation of the principles of knowledge and of the progress of reason, from sense to science and philosophy, 1794

> *And love Creation's final law*
> *Tho' Nature, red in tooth and claw*
>
> Alfred Tennyson, In Memoriam A.H.H., 1849

> *The powerful retractile talons of the falcon- and the cat-tribes ...*
> *survived longest which had the greatest facilities for seizing their prey.*
>
> Alfred Russel Wallace. On the Tendency of Varieties to Depart Indefinitely From the Original Type, 1858

Teeth, claws, talons, nails – there are gentler designs of Mother Nature that are also more important for survival of her children.

The mortality rate for all animals is the highest before they reach maturity. If you are a bird or mammal, your survival depends 100% on the care of your parents. Not enough mummy's milk – you are dead long before you learn what your talons are for.

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But it is not easy to figure out how Nature managed the *simultaneous* evolution of several *intrinsically uncorrelated* (?) functions, e.g., physiology + psychology of a kitten and then of the *same* animal in the role a mother cat where everything must function *in concert.* (Darwin's appeal to gradualism of evolution with no support by detailed/quantitative analysis of data is hardly acceptable for more than two "elementary functions".[44])

But talons or no talons, Wallace was a great (the greatest?) 19th-century naturalist. He collected more than 100 000 specimens in Malaysia and Indonesia and discovered more than a thousand new species, e.g., *Wallace's flying frog*.

As much as Darwin, he had been thinking on how and why species transform into new species and why the separation between different species is rather sharp.[45] He arrived at the same (but not quite) natural selection theory as Darwin, but, apparently, he was more ecologically minded than Darwin. A witness to that is Wallace's self-regulation principle in animal populations:

> *The action of this principle is exactly like that of the centrifugal governor of the steam engine, which checks and corrects any irregularities almost before they become evident; and in like manner no unbalanced deficiency in the animal kingdom can ever reach any conspicuous magnitude, because it would make itself felt at the very first step, by rendering existence difficult and extinction almost sure soon to follow.*

[44]An outcome of any kind of such analysis would depend, in particular, on the comparative size of atoms and molecules versus cells, for example.

[45]This sharpness, (that is, *disjointness*) of the corresponding *attractors in the dynamics of evolution*, probably, is determined by a presence of feedback loops in this dynamics. Negative feedbacks constrain the spread of attractors (that is, the intraspecies variation), whereas the positive feedback loops make different attractors (representing different species) drift apart.

I do not know if either he or Darwin realized that the *self-regulated* (negative feed-back) equilibrium can be oscillatory as in the *Lotka–Volterra equation*.[46] But Wallace gave a devastating analysis of how the ecology of St. Helena island (famous for Napoleon who had *not* studied it) was interfered with by European colonists and what happened to the equilibrium afterwards.

Disagreeing with Darwin, Wallace rejected the superficial similarity between artificial selection and natural selection in the wild:

> *... varieties produced in a state of domesticity are more or less unstable, and often have a tendency, if left to themselves, to return to the normal form of the parent species; and this instability is considered to be a distinctive peculiarity of all varieties, even of those occurring among wild animals in a state of nature, and to constitute a provision for preserving unchanged the originally created distinct species.*

The current view (if I get it right) is that *instability of domesticity* and intraspecies variation in general are mainly due (besides ever-present Gaussian bell-shaped variations[47]) to genome *crossover recombination* during meiosis that, unlike mutation, is (quasi)reversible.

Wallace, who could not know anything of this, gave a 19th-century explanation:

> *If turned wild on the pampas, such* [domesticated] *animals would probably soon become extinct, or under favorable circumstances might each lose those extreme* [artificially selected] *qualities which would never be called into action, and in a few generations would revert to a common type.*

Isn't it amazing that the two explanations have hardly anything in common?

The first one was unthinkable a hundred and fifty years ago. It depends on cellular/molecular data that have been obtained by many technically involved experiments the outcomes of which were by no means predictable a priori.

On the other hand, Wallace's common sense argument could have come from Lamarck with *adaptation* instead of *selection* (implicit in the above quote) and even from Aristotle whose choice of words would be *openly* teleological.[48]

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[46] This differential equation was used as a model of the predator-prey systems in the mid 1920s by the mathematical chemist Alfred Lotka and independently by the mathematician Vito Volterra. Prior to that, a similar equation was introduced in 1838 by the mathematician Pierre Verhulst for describing the number of individuals that an environment can support. Even earlier, in 1766, Daniel Bernoulli, stimulated by the inoculation controversy, solved an equation of this type in his study of smallpox epidemics.

[47] These variations, as was pointed out by Schrödinger and ≈ 150 years earlier by James Hutton, may be purely phenotypical; thus, *non-inheritable*, contrary to one of the main premises of pre-Weismann Darwinism.

[48] Aristotle categorized "explanatory causes" into four classes:

> *the material cause, the formal cause, the efficient/moving cause*, and *the final cause*,

(Teleology is built into human language and it inadvertently pops up in our reasoning when we least expect it, especially in evolutionary biology and in psychology.[49] Even Darwin, who fought teleological thinking in biology for years, writes:

> *Natural selection cannot possibly produce any modification in a species exclusively for the good of another species.*

By logic, every explanatory non-teleological sentence about evolution with *for* in it must be either vacuous or self contradictory.)

Wallace was also skeptical about Darwin's powerful idea of *sexual selection*. He, apparently, found it too powerful. You can explain almost everything with it:

> *a certain feature evolves because the opposite sex happened to like it.*
> (*This dispenses with the necessity of reflection* – Poincaré would say.)

Thus, for example, Wallace identified the true role of what is now called *warning coloration in animals*, that Darwin originally attributed to sexual selection.

An essential point of divergence between Wallace and Darwin was in the dominant role of selection, in particular of sexual selection, in the *human* evolution that took merely 200 000–2 000 000 years depending on from where you count.

> *Man was generated from all sorts of animals,*
> *since all the rest can quickly get food for themselves,*
> *but man alone requires careful feeding for a long time;*
> *such a being at the beginning could not have preserved his existence.*[50]

It is hard to disagree that sexual selection *is* the most probable cause of, say, sophisticated courtship of birds – just think of a peacock burdened by his

---

where the latter corresponds to something like our *teleological*. It would be presumptuous however, even to raise it as a *question* what *exactly* Aristotle meant by these "causes" and what would be his vision on their respective roles in the performance of evolution; yet, this classification is inspiring anyway. For example, it brings the modern stochastic genome dynamics perspective on evolution to the same "material/efficient category" as Lamarckian ideas on means and causes of evolutionary modifications of organisms whereas "explanations by selection" go to the "formal category".

[49] Rephrasing von Brücke and/or Haldane,

> *Teleology is like a mistress to an evolutionary biologist: he cannot live without her but he's apprehensive of being caught up in her company by a mathematical physicist.*

Ernst Wilhelm Ritter von Brücke (1819–1892), as much as Claude Bernard and Hermann Helmholtz, can be regarded as a father of modern physiology. Among many other things, he studied the change of color in chameleons and the ways sounds of European and Oriental languages are produced.

John Burdon Sanderson Haldane (1892–1964) was one of the founders, along with Ronald Fisher and Sewall Wright, of *population genetics* – that is Mendelian dynamics in a *micro-evolutionary setting.* His idea (1941) of checking *semiconservative mode of DNA replication* with the use of ^{15}N (the heavy isotope of nitrogen), was implemented in 1957 by Matthew Meselson and Franklin Stahl in one of the most logically beautiful experiments in biology.

[50] Attributed by (Pseudo?)-Plutarch (2nd century?) to Anaximander, 610–546 BCE.

"useless" tail. Yet, only (adult?) males may be expendable; precarious childbirth and delayed maturity seems too high a price for the big brain in humans.

Apparently, the brain + language evolution went through a positive feedback loop. Such loops, be they positive or negative, must be abundant in the organisms/environment systems since organisms modify/shape environment. (The interactions between opposite sexes of the same species is a basic instance of that, where positive loops are more easily recognizable as they enhance sexual selection.[51])

An obvious environment of an individual in the early human brain evolution, besides one's mates, was one's *tribe/clan identified by common language*:

in a community of speakers and listeners,
selection favors the most articulate ones.

Besides, these tribes themselves became *units of selection*, where evolution speeds up by a significant factor for a population that is divided into N competing tightly knit groups. (*A little mathematics of the multiplication table type* suggests this factor may be almost as large as N but I am not certain about it.)

Wallace himself maintained that there must be *something transcendental* responsible for the emergence of higher cognition in humans. This does not look *scientific*, unless *transcendental* is read as

a simple yet subtle abstract structure that underlies human cognition and that is evolutionarily accessible.

A remarkable instance of this kind of a structure – *imprinting in young animals* – was described in an 1872 paper by Douglas Spalding. This marked the birth of psychology as a *science* separated from neurophysiology and ... Spalding's results shared the fate of those by Mendel of being ignored for several decades.

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An attractive feature of this structure for a theoretically inclined *Naive Mathematician* as well as for practically minded Mother Nature with her unsophisticated

[51] *Sexual preference for such developments in the female, must thus advance together, and so long as the process is unchecked by severe counterselection, will advance ... in geometric progression.*
Ronald Fisher

strategy of evolution by selection, is the *universality* of imprinting: A baby animal (of a certain class of species) takes the first moving object, *whatever it is*, for its mother. (Does this very simplicity of imprinting disclosed by Spalding's experiments that excites mathematicians hold it in low esteem among psychologists?)

Possibly, many (all?) fundamental patterns/units of human/animal psychology/behaviour are comparably mathematically simple/universal; thus, evolutionarily accessible. But they may have no apparently straightforward manifestations and be hard to detect by direct experiments.[52]

> *Les développements de la vie, la succession de ses formes,*
> *la détermination précise de celles qui ont paru les premières, la*
> *naissance simultanée de certaines espèces, leur destruction graduelle,*
> *nous instruiraient peut-être autant sur l'essence de l'organisme.*
>
> Georges Cuvier, Recherches sur les ossemens fossiles des quadrupèdes, 1812

Cuvier was a magician of palaeontology and comparative anatomy:

> *... after inspecting a single bone, one* [he modestly speaks of himself] *can often determine the class, and sometimes even the genus of the animal to which it belonged, above all if that bone belonged to the head or the limbs. ... This is because the number, direction, and shape of the bones that compose each part of an animal's body are always in a necessary relation to all the other parts, in such a way that – up to a point – one can infer the whole from any one of them and vice versa.*

Cuvier's analysis of the fossil data was the primary source of 19th-century evolutionary theories, but Cuvier rejected the idea of gradual transmutation of species proposed by Lamarck, since fossil records indicated abrupt rather than gradual changes, and he scorned the mechanisms of evolution suggested by Lamarck.

Would Cuvier accept natural selection as a scientifically valid solution to the problem of evolution? Would Huxley – *Darwin's bulldog* as he was called – stand a chance against arguments that Cuvier could master?[53]

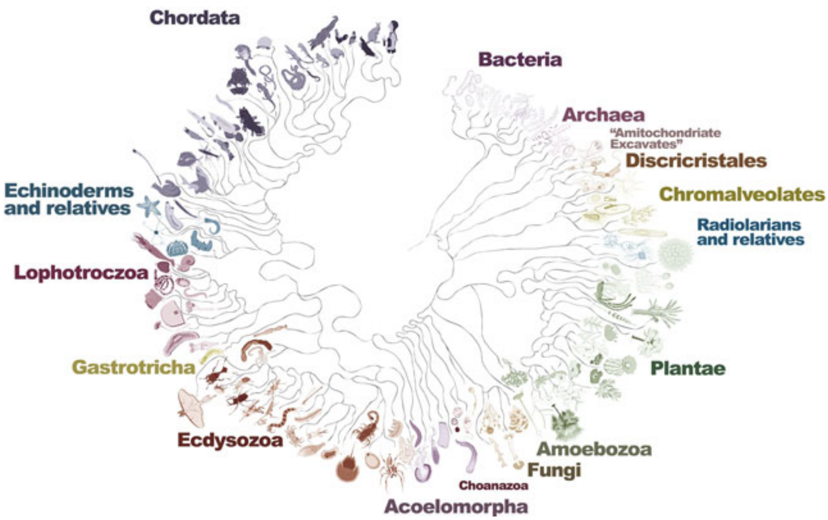
(Thomas Henry Huxley, probably, the second great comparative anatomist after Cuvier, beat bishop Wilberforce in a public evolution debate. But that was rhetoric, not science.)

[52] Another such pattern, *Hawk/Goose effect*, is a habituatory response of a baby animal who learns not to fear *frequently* observed shapes sliding overhead.

The structure of *imprinting* and of *Hawk/Goose* does not look simple if you go to the bottom of it. To see the problem, try to honestly describe these in *brain language* – in terms of retinal images and/or of properties of flows of signals received by the vision processing centres in the brain.

[53] Cuvier would have no problem with the (pre-Darwinian) concept of *purifying selection*, that is similar to weathering that smoothes out the shapes of old mountains and hills, but *creative power of selection in formation of species* would sound to him as bizzare as the *creative power of erosion in the formation of mountains.*

The discontinuity of fossil records that puzzled Cuvier remains a puzzle and his idea of the essential role of *catastrophes* in shaping evolution may be correct. Who knows, if not for the catastrophes, if the Earth were moving through time without hitting potholes and bumps on the way – nice, smooth, and continuous ride – then Nature would be more than satisfied with Precambrian jelly fish – there would be no need for the luxury of "higher" plants, animals, and people.



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Les végétaux sont des corps organisés vivants,
jamais irritables dans leurs parties, ne digérant point,
et ne se mouvant ni par volonté, ni par irritabilité excessive.

Jean-Baptiste Lamarck, Philosophie Zoologique, 1809

A definition worth its name – mathematicians learned this from Alexander Grothendieck – is not a concise wording of what everyone knows, but a pointer toward unknown. Many unexpected fruits have been harvested from what grew from the seeds of ideas in his definitions.

Lamarck had never read Grothendieck. Not only did his definition miss moving carnivorous plants, such as *the Venus Flytrap* (undoubtedly known to Lamarck), but his criteria are inapplicable to most of the tree of life. The two minor branches of this tree – plants and animals – are poor representatives of (mostly unicellular) life on Earth.

The tragedy of Lamarck was that he put wrong signs at the portals of the edifice of the evolution theory he had erected. Like Columbus, who misread *America* for *Indies* on the coastline of the new land, Lamarck misspelled *selection* for *adaptation*.

Worse than that, unlike his evolutionary successors, Lamarck had proposed several biological mechanisms underlying evolution. His ideas had the "drawback" of having scientific (*material* and/or *moving* in Aristotelean terms) ingredients in them that made these ideas experimentally verifiable; they turned out to be (essentially but not fully) incorrect.[54]

If you take for granted the miracle of adaptation of organisms to environment during their lifetimes, you may equally accept the Lamarckian idea of adaptation on the evolutionary time scale via some non-teleological(!) mechanism that allows an environmentally channeled influence of the potential future on certain internal processes in an organism that could benefit descendants of this organism.

This idea creates no logical problem in many instances where *selection* and *adaptation* are interchangeable in "explaining" evolution; yet, as Darwin points out, no specific environmentally induced mechanism (e.g., what was suggested by Lamarck) seems plausible in *evolution of the mode of reproduction* – this most essential feature of any species. (An instance of such "non-Lamarckian" feature is the *stability of* 1 : 1 *sex ratio* for such species as elephant seals with harems of several dozen females.) Another "non-Lamarckian picture" is seen, as was also indicated by Darwin, in *evolution of social insects*. (There is no essential disagreement, however, between Lamarck and Darwin about the folk idea on inheritance of acquired traits.)

Geometrically speaking, the connectivity of LIFE ON EARTH along the time coordinate via the heredity threads between organisms is rather tenuous unlike the full-fledged spacial connectivity/unity of individual organisms.

But is it possible *to directly* disprove, Lamarck's idea of adaptation and show that beneficial mutations take place *prior* to the change in the environment where they turn out to be beneficial? You cannot do this by staring at fossilized remnants of extinct animals.

Yet, in 1943, Salvador Luria (biologist) and Max Delbruck (physicist) thought of an experiment the logic and the beauty of which would delight Gregor Mendel. This goes, roughly, as follow.

Grow a colony with, say, about a billion bacteria, starting from a single cell, e.g., of E. coli, a bacterium adored by bacteriologists. (E. coli colonizes your guts within several hours of birth, where it switches from aerobic to anaerobic life and it adheres to the mucus of your large intestine until you die.)

Suppose, that when we apply some factor X to such a colony, say a virus (e.g., Bacteriophage T1 – Escherichia coli's best friend) which is deadly for the cells, there is, nevertheless, a small probability, say $p_1 = 0.02$, that some cell in the colony survives. In fact, such a survivor is a result of a mutation that brings a certain new property Π to the mutated cell that is manifested in tolerance of the mutant to the X factor. Moreover, and this is essential, Π passes on to all descendants of our Π-cell.

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[54] Some Darwinists rejoice in finding inconsistencies in Lamarck as if this makes their own theories more substantial.

If you apply this procedure, to, say, one thousand different colonies, then about twenty will have survivors in them. Discard dead colonies and look how many among the remaining twenty have *two* (or more) surviving bacteria. There are two conjectural alternatives.

1. *Lamarckian Adaptation.* If property Π develops in response to the factor X *after* it has been introduced, then the probability of having two survivors will be $p_2(\text{adap}) = 0.0004 = p_1^2$ being the results of simultaneous occurrence of two *independent* p_1-events. In this case

a presence of three colonies with two survivors is highly unlikely.

2. *Sheer Luck.* If some bacteria mutate *prior* to introduction of X, then this could happen before the last round of division with *half* billion bacteria on the plate with probability $0.01 = p_1/2$. This makes $p_2(luck) \geq 0.01$ and there must be something like

from seven to thirteen colonies with two survivors in them.

This is what Claude Bernard calls *active science* where *you make experiments not to confirm your ideas, but to control them*.

When you check what actually happened, you find out that, unequivocally, **2** is true and **1** is false: About half of not fully dead colonies have two or more active bacteria in them.

(To Lamarck this would be no more in a contradiction with his idea of evolution, than, say, a guillotined head could be a counterexample to how the use/disuse of an organ implies an adaptive evolution of this organ in a *slowly* changing environment.)

Stop! How can you tell how many bacteria out of *billion* are alive?

Elementary, my dear Watson. Shake your cells in a liquid to randomly mix them and spread the culture on a nutrient plate. (Technically it is better first to spread the fully alive culture and then apply X.) Then each survivor – each Π-cell – starts dividing and in a short while you see as many colonies on the plate as you had Π-cells to start with.

(If we do not introduce X and keep the colony alive at a limited size with a constant flow of nutrients, then, some time afterwards, only a few cells will have surviving descendents and, eventually, only one cell. This follows from Anaximander's **Theorem 1** when it is empowered by an estimate of the *probability of extinction* that was calculated by Francis Galton and Henry William Watson in their 1874 paper. There is hardly one chance in a million that the Π-cells from the original colony would have descendants in a few years.

The *original* resistance to X is not preserved[55] and none is visible in the colony; yet, the "dormant guardians" are there and they "fight back" when the

[55] It can be preserved, if the colony was exposed to the virus: CRISPR-based adaptive immunity in bacteria is heritable. This justifies Lamarckian ideas if these are applied to evolution of immune systems in prokaryotes.

colony is attacked by X. Lamarck would say that colonies do adapt by "voting for survival" of their luckiest members and that a female egg-cell might be similarly selecting the fittest contender out of hundreds of millions of available spermatozoids; thus, ensuring adaptation of the progeny. Let Darwin and Weismann disprove this.)

Another idea of Lamarck is the presence of a *complexifying force* that *drives* organisms from simple to complex forms with environmental "pressure" being superimposed on this force. You cannot discard this by brandishing the banner of natural selection in the air, so try something cleverer.

Probably, such a "force", if it exists, resides in the *logic of chance* in the evolutionary game (frame?) of LIFE, possibly, expressible in a language of a Grothendieck-like mathematics. But deciding one way or the other seems more difficult, than, for example, (quasi)*rigorously* "deriving" irreversibility in thermodynamics from the reversible laws of physics where the present day mathematical rendition of Boltzmann's arguments does not seem satisfactory.

In either case *mathematics of the multiplication table type* that suffices for the Luria–Delbruck experiment seems of little help to you. (But maybe the main problem is that we do not understand *multiplication table*.)

> *... there perished many a stock, unable*
> *By propagation to forge a progeny.*
> *For whatsoever creatures thou beholdest*
> *Breathing the breath of life, the same have been*
> *Even from their earliest age preserved alive*
> *By cunning, or by valour, or at least*
> *By speed of foot or wing.*
>
> TITUS LUCRETIUS, ON THE NATURE OF THINGS, 50 BCE(?)

> *... et ces espèces que nous voyons aujourd'hui ne sont que*
> *la plus petite partie de ce qu'un destin aveugle avait produit.*
>
> PIERRE-LOUIS MAUPERTUIS, ESSAI DE COSMOLOGIE, 1750

> *... just in proportion as this process of extermination*
> *has acted on an enormous scale, so must the number of*
> *intermediate varieties, which have formerly existed,*
> *be truly enormous.*
>
> CHARLES DARWIN, ORIGIN OF SPECIES, 1859

Dramatic cut-off of exponentially growing functions in a bounded space, called by Darwin *natural selection*, is an apparent logical necessity, not an intrinsically biological property of living systems.

The principle of natural selection no more "explains" evolution than differential equations "explain" mechanical motions, but this principle provides a conceptual framework and suggests a language for possible mathematical models of evolution.

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This must have been obvious to Maupertuis and to anybody who had ever pondered on population growth and who could fathom the immensity of $\exp T$, e.g., to Buffon, who along with Maupertuis, was wondering

> *What purposes then are served by this immense train of generations, this profusion of germs, many thousands of which are abortive for the one that is brought to life?*

to Benjamin Franklin, who wrote on some occasion in 1751:

> *no bound to the prolific nature of plants or animals, but what is made by their crowding and interfering with each others means of subsistence,*

to Euler, who, in *Introductio in analysin infinitorum* (1748), illustrates the exponential function by examples from population dynamics, and to Fibonacci who savors numerical subtleties of an idealized rabbit replication model in his *Liber Abaci* (1202).

An imbalance between the growth of the population and of resources also must have been known for centuries. The problem and an approach to it are discussed, for example, in Giovanni Botero's treatise *Delle cause della grandezza delle città* (1588).

A humanistic perspective on the solution of the overpopulation problem was proposed, in almost modern terms, by Nicolas de Condorcet in *Esquisse d'un tableau historique des progrès de l'esprit humain* that was published posthumously in 1795. But there is nothing humanistic in how Nature has been handling the (over)population problems for millions of years.

Condorcet and Malthus

Condorcet is remembered by mathematicians for his 1785 essay

> *sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*

where he proves his *jury theorem* on probability of a group of individuals to arrive at a correct decision and where he analyses the (*voting*) *paradox* of non-transitivity of the *order relation on decisions* defined by collective preferences.

His passionate manifesto

Esquisse d'un tableau historique des progrès de l'esprit humain

was written between October 1793 and March 1794, when Condorcet was in hiding having been condemned to death by the Robespierre government for his humanistic political views.

(About a century afterwards, a monument was erected in Paris with the torso of Lavoisier, who was guillotined forty days after Condorcet was found dead in his jail cell, and the head of Condorcet. This was not done on purpose.)

Thomas Malthus, in *An Essay on the Principle of Population*, published in 1798, sides with Nature in her solution to the overpopulation problem and expresses his skepticism on the social practicability of the solution proposed by Condorcet.

Malthus' influence is due to two facts.

1. In the 19th century, the number of readers who were willing to pay for a book with $\exp T$ in it reached the critical mass needed to make the publication of such a book profitable.
2. Darwin and Wallace were among these readers.

But the idea that technological progress can compensate for Malthusian $\exp T$ remains as much in contradiction with the multiplication table today as it was at the time of Malthus. The rules of arithmetic do not change, at least not on so short a time scale.

Entropy versus Energy Barriers in the Development of Science

The world is full of magical things patiently
waiting for our wits to grow sharper.

Bertrand Russell

There is a fundamental difference in the logical structure of principles of physics, such as *Newton's second law*, and those of evolutionary biology, such as *adaptation by selection*, that is reflected in our perception of these principles.

Learning and understanding physics is hard; one needs a firm mathematical background and a non-trivial intellectual effort to comprehend the meaning and the consequences of something like the second law, for example. Only a small percent of "educated people" even understand what it means to UNDERSTAND the second law.[56]

On the other hand the idea of *evolutionary adaptation by selection* strikes anybody, who is aware of the enormousness of exponential growth, as obvious.

[56] This is why the members of *flat Earth societies* do not argue against the second law.

(Realizing insufficiency of this idea in its original Darwinian form needs a higher level of mathematical sophistication.[57])

But why hadn't the full-fledged idea of evolution by selection sprouted from the brilliant mind of some mathematician such as Daniel Bernoulli or Leonhard Euler?

An idea in science, as a molecule in a chemical reaction, in order to reach the critical maturation point may need not to climb over a mountain of high energy, but rather to follow a narrow pathway in a labyrinth of tunnels through this mountain. Once such a pathway is found, everybody can follow effortlessly.

There may yet be other simple ideas in science waiting to be discovered that are as illuminating as the idea of the pivotal role of the exponential function in evolutionary dynamics. If you think such ideas are "obvious", pinpoint a single one of them.

> *... preservation of favorable variations, and the destruction of injurious variations, I call Natural Selection, or the Survival of the Fittest.*
>
> Charles Darwin, Origin of Species 5th edn. 1869

It is not this survival rhetoric however that had been mainly occupying Darwin's mind but a persistent idea of turning the *cut-off effect on the exponential growth in biology* into a *principle of selection* (of *continuity*?), something like Maupertuis' *principle of least action*.[58]

Maupertuis, who had been working over his principle for two decades, believed that Nature always minimizes/optimizes whatever she does and he must have tried to figure out the formula for "evolutionary action" that is minimized by Nature in the course of selection. Part of this must be *time*, he would assume, since not so much *perfection* of your fitness makes you the winner, but how fast, lets imperfect, your fitness can be evolutionarily achieved. But he, probably, could not guess and write down other terms in this "action".

Besides, Maupertuis would observe, that mutability and the reproduction rate must be sufficiently high in order for *evolution by selection and nothing else* to be logically/mathematically feasible.

Very roughly, R^T for R being the reproduction rate, must "beat" something like 2^P where P is the number of "variable parts" of an organism and where a presence of selection, that channels evolution by pruning off the majority of mathematically conceivable branches of development, makes the estimated time T needed for evolution longer rather than shorter.

On the other hand, the mutation rate cannot be too high; otherwise, deleterious mutations, that are in a dominant majority, would result in extinction. The

[57] On the other hand, mathematically insensitive people may be driven against the very ideas of evolution and natural selection by some (Freudian?) psycho-sociological mechanism.

[58] Maupertuis identified a certain quantity, called *action* $A = A(motion)$, such that a *moving* physical system S minimizes this A, or rather, the *motion* of S, thought of as a spatial curve, satisfies the corresponding *Lagrange–Euler* (*differential*) *equations*. Lagrangian and Hamiltonian incarnations of this principle are present in all branches of mathematical physics.

most dangerous are *dormantly deleterious mutations*, e.g., those that increase the mutation rate).

Being a student of Newtonian (non-Aristotelian) mechanics (with impulses as coordinates), Maupertuis would assume that the main *biological observable* (feature) that evolves by selection is "mutability", or rather its opposite – *fidelity of reproduction*. (All Nature knows is to keep the mutation rate as low as she possibly can – evolution is a random dance on the razor's edge between stagnation and extinction.[59])

Finally, Maupertuis, a mathematical physicist rather than a pure mathematician, would try to match rough numerical estimates of this kind with the schedule of the fossil data.

Well ... Maupertuis, had done none of this and neither had Darwin who remained dissatisfied with the selection theory of evolution, judging by how insistently he was *convincing* himself that natural selection explains evolution and by how eloquently he was declaring that he came to *believe* in the validity of the idea. It is for myths, not for science, to *explain* the world and life and it is up to preachers and politicians, not for scientists, to *convince* people of anything.

Darwin wanted more of his selection theory than being merely a *convincing explanation* of the evolution of Life on Earth but mathematics of the 19th century had nothing to offer for substantiating his vision.

A cautious 21st-century mathematician would not expect a single clear cut mathematical theorem/theory fulfilling Darwin's dream but he/she may still envisage a possibly vague, yet, *truly mathematical* setting embracing the idea of LIFE.

An instance of such a not quite precise but purely mathematical idea often used in physics is that of representing a state of a dissipative dynamical system, e.g., of a quasi-stationary flow of a viscous fluid, as an *ensemble of attractors* in the corresponding phase space.

But no such concept, nothing of what one expects to find in 20th-century dynamics, seems suitable for expressing much of the whole idea of LIFE.

(There may be dynamic-theoretic models of some of LIFE'S fragments. For instance, one can possibly describe the multi-scale time feature of *evolution-by-selection*, with different classes of *units of selection*[60] on different time scales, in the language of "multistage dynamical systems" where attractors themselves are subject to dissipative-like *predominantly contractive* dynamics.)

---

[59] Nature faces the same problem as a public education system – it should be *almost but not fully* perfect: If students 100% comply with the demands of their teachers, society stagnates; but too little rigour results in *error catastrophe*, cancer of corruption and extinction.

What is amazing (for non-Lamarckians) is that besides sheer sloppiness, there are provisions by Nature for (quasi-random) "desirable mutations", e.g., in specified loci of genomes of parasites evading the immune systems of their hosts and in genomes of bacteria under stress.

[60] Mathematically speaking, units of selections are *particular* biological observables but it is unclear *particular* in what sense. For example, can *fidelity of reproduction* – one of the basic and evolutionarily slowest observables – be taken for such a "unit"?

It seems that a satisfactory understanding of genomes' stochastic dynamics and transforming the mathematical poetry of the Darwinian idea of evolution by selection to a hard-boiled scientific theory lies a long way off.

> *... our* [19th] *century will be called ... the century of the mechanical view of nature, the century of Darwin.*
>
> LUDWIG BOLTZMANN

The abstractly logical – *formal* in the Aristotelian sense – Darwinian SELECTION PRINCIPLE may be alluring to mathematicians and to *mathematical* physicists like Boltzmann; yet, this *formality* is disconcerting. How can some property of the logic/syntax/mathematics of the language that one uses stand for the *main reason* of something *physically* or biologically occurring?

Refined mathematical concepts may seem no more helpful for understanding the harsh truths of the real world than poetic metaphors for this purpose. Only *Naive Mathematician* may take, for instance, the summation rules of infinite series for "explanation" of *Zeno's paradoxes* or the general theory of (classical or/and quantum) Einstein–Lorentz spaces for a full physical model of SPACE-TIME. But *Naive Mathematician* may protest by pointing out, for example, that when Laplace writes about

> *Une intelligence qui, pour un instant donné, connaîtrait toutes les forces ... embrasserait dans la même formule les mouvements des plus grands corps de l'univers et ceux du plus léger atome ....*

he hardly thinks that the solution of the problem of determinism may be advanced by a study of the mind of this metaphoric *intelligence* by an experimental neuropsychologist (as some 21st-century thinkers suggest), but he would accept that the (quasi)deterministic behaviour of planetary motions might be explicable in the light of *unique solvability of differential equations* and/or KAM-like theorems.

The skepticism of the hard core scientists should not divert us from a quest for mathematics that would bring "light of sense" to evolutionary biology.

We want to have maximally general, yet biologically sound mathematical concepts of

gene, organism, population, unit of selection, environment, adaptation

so that we can make sense of the following questions.

- What are probability (?) spaces where selection works?
- What defines (dis)connectivity of genomes, organisms, populations?
- Why are adaptations represented by (quasi)fixed points of (gradient?) evolutionary dynamics in *different* spaces, e.g., of genes in the environments of other genes and of organisms themselves in the "natural" environments, leads to *similarly looking* "coarsely grained adaptation" of populations, as we saw in the example of domesticated species reverting to wild type?

8. Brain

Tell me where is fancies bred, Or in the heart or in the head.

Shakespeare, The Merchant of Venice
(Probably, written between 1596 and 1598)

... mental activities are entirely due to the behavior of nerve cells, glial cells, and the atoms, ions, and molecules that make them up and influence them.

Frances Crick, The Astonishing Hypothesis(1994)[61]

Shakespeare would not have been intimidated by the scientifically sounding "*mental activities*" instead of his "*fancies*"[62] and *the Hypothesis* would not have stricken him as especially astonishing.

After all, it has been known since about 2500 BCE that different types of head injury produce different symptoms, as it was recorded in *Edwin Smith Surgical Papyrus*, of about 1500 BCE where the idea of Brain appeared for the first time.[63]

Shakespeare could not have been acquainted with the *Surgical Papyrus* – this was discovered in the middle 1800s, but he might have been aware of the following.

The seat of sensations is in the brain. ...
All the senses are connected in some way with the brain ...
This power of the brain to synthesize sensations
makes it also the seat of thought.

[Attributed to] Alcmaeon of Croton (≈ 450 BCE)

the source of our pleasure, merriment, laughter, and amusement
as of our grief, pain, anxiety, and tears, is none other than the brain.

Hippocrates (?), On the Sacred Disease, (≈ 425 BCE)

---

[61] In the early 1950s, Frances Crick and James Watson worked out, with the X-ray crystallography data contributed by Rosalind Franklin, a correct helical model of DNA. (An incorrect triple-stranded helix structure was earlier suggested by Linus Pauling). They concluded:

it therefore seems likely that the precise sequence of the bases is the code that carries the genetical information.

Crick also proposed The central dogma of molecular biology: DNA→RNA→Proteins.

This *... deals with the detailed residue-by-residue transfer of sequential information. It states that such information cannot be transferred back from protein to either protein or nucleic acid.*

From age 60 until his death (2004) Crick worked on the Brain; in particular, he tried to identify specific neuronal processes responsible for consciousness.

[62] No matter how hard you try, you cannot say much about the arrow [brain]⇝[mind] without resorting to metaphors. (Poetic "*bred*" prompts in you a surge of ideas, while dry "*entirely due*" transmits zero positive information.)

[63] Surgical Papyrus is an incomplete copy of a text attributed to *Imhotep* (≈ 2650–2600 BCE), called "*inventor of healing*", who was a high priest of the Old Kingdom of Egypt as well as chief builder and chief physician in his time. He is the first physician known by name.

... sense-perception in sanguineous animals is the region of the heart ...

Aristotle, Sleep and Sleeplessness ($\approx$ 350 BCE)

It is hard to believe that clinical observations by Alcmaeon and Hippocrates did not convince Aristotle, who had not accepted "the Astonishing Hypothesis" because:

(1) the heart, unlike the brain, connects with all the sense organs;[64]
(2) the heart is more centrally placed;
(3) the heart in embryos develops before the brain;
(4) invertebrates, who have hearts but no brains, have sensations;
(5) the heart but not the brain is affected by emotions;
(6) the heart is warm but the brain is cold;
(7) the heart but not the brain is essential for life.

Half a century later, Alexandria's anatomist Herophilus (335–280 BCE) discovered that nerves spread from the brain throughout the body in agreement with the idea that the brain was the controlling organ in Man.

His younger colleague Erasistratus (304–250 BCE), who believed that the psychic pneuma was transmitted through motor nerves to muscles, appreciated the separate neural pathways for motor and sensory functions. He also suggested that the degree of intelligence in animals is correlated with how much cerebral hemispheres are convoluted.

But the experimental evidence for the brain control over the body came four centuries later, when Galen of Pergamon (129–200?) had shown that cutting the recurrent laryngeal nerves which innervate the larynx makes a pig stop squealing but not struggling.[65]

Here is his reaction to those who were not convinced of the dominance of the brain after having attended this experiment.

> *When I heard this, I left them and went off, saying only that I was mistaken in not realizing that I was coming to meet boorish skeptics; otherwise I should not have come.*

Ancients had no idea of a cell and could not fathom how the brain works. And when the cellular structure of the brain started unravelling itself in all its beauty, founders of *neuronal theory* were inspired to speak of the brain in poetic language:

> *The brain is waking and with it the mind is returning.*
> *It is as if the Milky Way entered upon some cosmic dance.*
> *Swiftly the head mass becomes an enchanted loom*

[64] Nerves are less prominent than blood vessels.

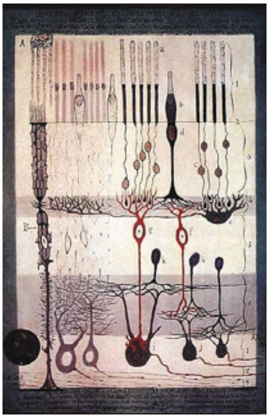
[65] Besides founding experimental medicine including experimental neurophysiology, Galen greatly advanced anatomy, physiology, pathology, and pharmacology of his time. For example, he showed that urine comes from the kidneys, he demonstrated that the larynx generates the voice and he recognized the essential difference between venous and arterial blood. He also developed several surgical techniques, including one for correcting cataracts.

where millions of flashing shuttles weave a dissolving pattern,
always a meaningful pattern though never an abiding one;
a shifting harmony of subpatterns.

Charles S. Sherrington, Man on his Nature, 1942

To know the brain . . . is equivalent to ascertaining the material course
of thought and will, to discovering the intimate history of life
in its perpetual duel with external forces.

Santiago Ramon y Cajal, Recuerdos de mi vida 1917



“Structure of the Mammalian Retina” © 1900 By Santiago Ramon y Cajal / Wikimedia Commons / Public Domain

Cajal and Sherrington, apparently, speak of *the human* brain but experimental neuroscience and, in part, brain anatomy relied on the study of animal brains, starting from ox and pig brains dissected by Galen and continued with a microscopic study of insect brains since the 18th century.

. . . the brain of an ant is one of the most marvelous
atoms of matter in the world, perhaps more so than
the brain of a man.

Charles Darwin (1859)

For instance, the collective mind of ants is able to achieve shortest paths between locations in a rugged terrain:

a busy ant highway between an anthill and a source of food usually implements a nearly shortest possibility.

Ants, Chemistry and Logic

There are more than 10 000 different species of ants and variations within a species may also be high. On the average (?), there are about a quarter of million brain cells in an ant. But there are ants and ants with their weights ranging from 0.01 mg to half a gram, where, in small ants, the brain makes more than 10% of the animal body weight – comparable to that in a newborn infant and by far more than 2% in adult humans.

If my opinions are the result of the chemical processes
going on in my brain, they are determined by the laws
of chemistry, not those of logic.

John Haldane, The Inequality of Man (1932)

Ants mark their trails with pheromones and themselves tend to choose the routes that have stronger pheromone odors. All things being equal,

the number of ants that pass back and forth on some track, say during 1 h, is inversely proportional to the length of this track;

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hence, the shortest track becomes the smelliest one, thus, eventually preferred by the ants.

What has made this algorithm evolutionarily attainable is its simplicity and universality. Probably, basic programs running within our minds, just in order to exist at all, must be comparably simple and universal. But there is not any "law" in sight[66] that determines ants' opinions of where to go.

Cajal and Sherrington

Cajal studied nervous and brain tissues with a *silver staining technique* discovered by Camillo Golgi in 1873 and established that the *neurones* are basic units of nervous structure. This theory was completed with the introduction of the concept of *synapse* by Sherrington who writes in his 1906 book The Integrative Action of the Nervous System:

> *At the nexus between cells if there be not actual confluence, there must be a surface of separation … In view, therefore, of the probable importance physiologically of this mode of nexus between neurone and neurone it is convenient to have a term for it. The term introduced has been synapse.*

[66] It is doubtful whether the expressions "*laws of chemistry*" and "*laws of logic*", unlike those of "*laws of classical mechanics*", make any sense at all.

9. Mind

The energy of the mind is the essence of life.

ARISTOTLE

As followers of natural science we know nothing of any relation between thoughts and the brain, except as a gross correlation in time and space.

CHARLES SHERRINGTON

WHAT IS THE MIND? WHAT ARE THOUGHTS?

How much have we to know about the brain to understand the mind?

Imagine that by the middle of the 21st century we will have learned as much about the brain as we know to-day about the fundamental quantum mechanical laws of matter and energy or about the nanoscale dynamics of the cell. Would it help?[67]

Or, will we only more clearly realize our helplessness and, following Pauli, will say that "mind", like "reality" is

> *something self-evidently known* ... [but it is] *an exceedingly difficult task ... to work on the construction of a new idea of* mind.[68]

But is it possible at all to translate the following poetic mage of brain/mind to the language of science?

> *The eye sends ... into the cell-and-fibre forest of the brain throughout the waking day continual rhythmic streams of tiny, individually evanescent, electrical potentials.*
>
> *This throbbing streaming crowd of electrified shifting points in the spongework of the brain bears no obvious semblance in space pattern, and even in temporal relation resembles but a little remotely the tiny two-dimensional upside-down picture of the outside world which the eyeball paints on the beginnings of its nerve fibers to the brain.*
>
> *But that little picture sets up an electrical storm. And that electrical storm so set up is one which affects a whole population of brain cells. Electrical charges having in themselves not the faintest elements*

[67] The molecular machinery of a living cell *transcribes + translates* each *codon* – triple of four basic nuclear acids in DNA (except for three stop codons) – to one out of twenty standard amino acids (with one stop codon sometimes coding for *Selenocysteine).* The particular coding/translation rule used by Nature, that is *the same for* (almost) *all* organisms on Earth and that is called the *genetic code*, is, for an experimentalist, the most fundamental law of biology. On the other hand, nothing (?) in life would visibly change if the code was somewhat different. What is essential in biology from a mathematician's perspective is the *principle* of coding, rather than the *specificity* of the code being used.

The brain may harbor an immensity of such "coding arbitrariness"; an accumulating avalanche of the detailed knowledge of this "code" may block out rather than promote our understanding of the arrow [brain] $\rightsquigarrow$ [mind].

[68] The idea of "reality" is inseparable from "mind": *Cogito ergo sum.*

of the visual – having, for instance, nothing of "distance," "right-side-upness," nor "vertical," nor "horizontal," nor "color," nor "brightness," nor "shadow," nor "roundness," nor "squareness," nor "contour," nor "transparency," nor "opacity," nor "near," nor "far," nor visual anything – yet conjure up all these.

A shower of little electrical leaks conjures up for me, when I look, the landscape; the castle on the height, or, when I look at him approaching, my friend's face, and how distant he is from me they tell me.

Taking their word for it, I go forward and my other senses confirm that he is there.

SHERRINGTON, MAN ON HIS NATURE

What does it mean to understand mind? What are the right questions to ask?

It seems there is an invisible interface between the brain and what we perceive as our mind – an interface that is comparable in its structural complexity to the *machinery of embryological development* that, as Thomas Hunt Morgan says, effectuates the transformation of what is "written" in the genes into [phenotypic] *characters that are used by the* [classical] *geneticist.*

What I cannot create, I do not understand.

RICHARD FEYNMAN

Here we speak of creating *abstract models* not necessarily implemented by physical devices,[69] and some of them may not need a knowledge of the brain, at least for modelling "parts" of the mind, such as *memory*, for instance.

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[69] Ability to create does not imply understanding. Animals (as well as plants and bacteria) can create their approximate copies but even the smartest of the human stock have only the vaguest images in their minds of the ways of embryological development of their progeny.

Memory . . . an Organ, as the Eye, Ear, or Nose,
where the Nerves from the other Senses concur and meet. . . .
. . . I conceive to be nothing else but a Repository
of Ideas formed partly by the Senses, but chiefly . . .
. . . receiving, and being excited by such Impressions,
they do again renew their former Impression . . . .

Robert Hooke, On Hypothetical Explication of Memory, Lecture to the Royal Society, 1682

This looks almost like a verbal description of a *mathematical model* of memory, but some do not like such an idea.

Intellectual snobbery makes them [mathematicians?] *feel they can produce results that are mathematically both deep and powerful and also apply to the brain.*

Frances Crick, What Mad Pursuit (1988)

However, new ideas on mind/brain came from the mathematical side along with the concepts of *information flows* and of *Turing machines* that allowed formulations of new kinds of questions, such as:

Can machines do what we (as thinking entities) can do?

Alan Turing, Computing Machinery and Intelligence (1950)

Attempts to disprove Turing's reasoning that a digital computer can imitate human intelligence are in line with the argument that people could not have come from the same stock as monkeys, since monkeys lack moral virtues and do not abide by traffic laws.

And it is not only apes and monkeys – also bacteria in our guts are our remote cousins as far as biology is concerned. The unity of life on Earth lies deeper then similarity in anatomy and physiology. Something of this kind must be true about "intelligence" except we do not know what it is.

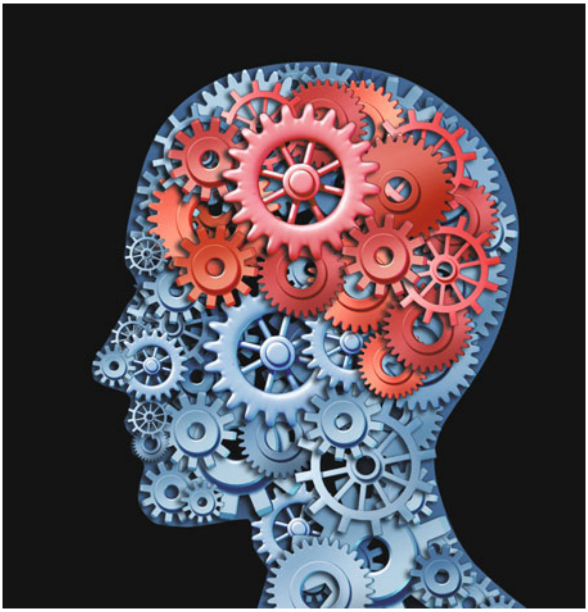
The difficulty in understanding the human mind is that it is *unique of its kind*; hence, there is no language to speak about it. The brain, on the contrary, is shared by almost all animals on this planet. This is the way neuroscience advances but the scientific theory of mind has stayed still for several millennia.

And how on Earth can one disprove Turing? We *are* biological machines as much as, say, ants are, albeit, individually, we have more neurons in our brains:

Man is so complicated a machine that it is impossible
to get a clear idea of the machine beforehand,
and hence impossible to define it.

Julien Offray de La Mettrie, Man a Machine. 1748

But maybe, what happens in our minds with wheels within wheels within wheels in them is too complicated to be representable by mere math? This is what Edgar Allan Poe thought in connection with chess.



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Arithmetical or algebraical calculations are, from their very nature, fixed and determinate . . . [But] *no one move in chess necessarily follows upon any one other.*

Edgar Allan Poe, Maelzel's Chess-Player, April 1836

Of course, Poe could not have known of Turing, but he was aware of the calculating machine of Babbage and argued that playing chess *cannot* be modelled on such a machine. And although what he was saying was formally incorrect, a fundamental difficulty[70] he pinpointed in such a modelling remains unresolved up to the present day.

Another problem with understanding human mind is that our intuitive concept of *intelligence* shaped by the *existential ego* of Cartesian *cogito ergo sum*, is buried in the multilayered cocoon of *teleology* – *purpose, function, usefulness, survival.* Turing purposefully avoids the question – *What is intelligence?* – since you cannot answer this question unless you develop an appropriate language. (It is the same as with the question – *What is* Life? – that cannot be answered in the language of Life on Earth.)

However, Turing suggests that a "human-like intelligence" can be built in a machine by subjecting it to education. Indeed, *learning* is a more intelligent

[70] As one tries to analyze a position many moves ahead, one becomes overwhelmed by the exponentially growing numbers of branches of possible game strategies.

concept than *intelligence*,[71] but one will not go far with it if one assumes that the child brain is *something like … lots of blank sheets.*

> *Instead of trying to produce a programme to simulate the adult mind, why not rather try to produce one which simulates the child's?*
>
> *If this then were subjected to an appropriate course of education one would obtain the adult brain.*
>
> *Presumably the child brain is something like a notebook as one buys it from the stationer's. Rather little mechanism, and lots of blank sheets.*
>
> Alan Turing, Computing machinery and intelligence

An unbridled delight in the brilliance of our "intelligence" blinds us to seeing the essence of the mind but modelling the learning process by a child may bring the light into the picture.

> *Man is most nearly himself when he achieves*
> *the seriousness of a child at play.*
>
> Heraclitus

All a child knows is play and this is how he/she learns. The mathematics of this *playful learning process*, that is a transformation of the flow of electric/chemical signals the brain receives into a coherent picture of *external world* in the first two years of human life, is as intricate and mysterious as that of emergence and evolution of *live structures*.

And it is, so to speak, *free learning*: if you subject a human (or animal) infant to "education" you only impair the learning, since you have no idea of what happens in the brain/mind of a child. (Not that you know better how your own mind works.) Applying your "adult intelligence" to aid an infant's development is like a use of forceps in helping an amoeba to divide. No surprise, learning programs that have been developed until today are a far cry from Turing's dream.

In order to progress, instead of inventing more and more "intelligent" definitions of *intelligence*, *we must recognize our ignorance* as Feynman says, forget/rectify[72] how we "naturally" see ourselves and start with potentially answerable questions.

[71] *Learning* brings along a relatively slow *time coordinate* in the description of *mind* similarly to how *evolution* brings such a coordinate in the picture of *Life on Earth* where *evolutionary learning* is described in the language of *natural selection*.

[72] *L'esprit humain se plie à une manière de voir* – Lavoisier says in his paper on *phlogiston* – the common sense idea of *warmth* disguised as a scientific concept in pre-Lavoisier chemistry.

If you follow Lavoisier's path of reasoning, you will find his words applicable in connection with "intelligence" and "consciousness" as well. These notions are helpful in practical psychology but no common-sense definition of *intelligence* and/or *consciousness* can be taken for anything *scientific* in Lavoisier's understanding of the idea of science.

What are the *logical* (*molecular-like*) *units*[73] that *abstract intelligence* is composed from?

An absence (presence?) of what structure(s) in the mind of a baby ape makes its ability and willingness to learn after the age ≈ 1.5 years to fall below that of a human child but still keeps its learning gradient above that of an adult human?

What is the mechanism(s) responsible for rarely occurring structurally elaborate patterns within "human intelligence" such as exceptional musical and mathematical abilities? Are there conterparts for these in animals?

Do these patterns, that are harmful rather than helpful for *survival in the wild*, come by way of natural selection and if so what is the pool of possibilities/variations of these patterns?

What is a non-teleological description of the role played by *predictable future* in the functioning of an "intelligent system"?

What is a simple/short list of questions (or, rather an interactive algorithm generating questions depending on the history of a conversation) that would be easily answerable by a human of any culture, e.g., by a Cro-Magnon child (IF the language problem is somehow taken care of), but would cause an insurmountable difficulty for currently available "imitation programs".[74]

We need to design arguments and/or experiments for finding the answers that would serve to control rather than to confirm our ideas, being most satisfied with an outcome if it does not resemble anything we had imagined. Thus we shall be able to trace and to seal the route by which the self-gratifying illusion of our "intelligence" came from and to start a walk on a road toward understanding what the mathematical essence of mind/learning/intelligence is.

Quite likely, a full instruction for making a human-like learning system can be written down on a couple of hundred pages.

But even if you are on the right track for doing so, it may be a long way to go. It took Nature half a billion years and quadrillions of tries to arrive at the present design of our nervous system and we may need quadrillions of man/computer hours for designing a comparable system ourselves.[75] And the detailed knowledge of the wiring of the human brain would not necessarily help because of the indecipherable stochastic redundancy in its architecture.

[73]These units must be functions of *abstract context/purpose free* "variables", like *imprinting*, that applies to [*first/second* ...] (moving object) and of *Hawk/Goose* that depends on [*frequently/rarely*] (occurring event).

[74]Such questions must refer to (and/or recycle) phrases that had been already used in the course of a dialog. Probably, the length L of the (suitably defined) *shortest naive* imitation program that is undetectable by a clever algorithm after an exchange by n sentences must grow at least exponentially, $L \sim 2^n$.

[75]One may accept this as a possibility if one believes that $NP \neq P$.

10. Mysteries Remain

Ignoramus et ignorabimus.

EMIL DU BOIS-REYMOND (1872)[76]

In mathematics there is no ignorabimus.

DAVID HILBERT (1900)

Wir müssen wissen – wir werden wissen!

DAVID HILBERT (1930)

© Henrik5000 / Getty Images / iStock

Du Bois-Reymond suggested[77] that, possibly, humans shall *never* understand the following:

1. *Nature of matter and force.*
2. *Origin of motion.*
3. *Origin of Life.*
4. *Emergence of seemingly purposeful organisation in Nature.*
5. *Generation of conscious sensations by unconscious nerves.*
6. *Source of intelligent thought and Language.*
7. *Non-determinism of free will in deterministic Universe.*

Do you agree with what Du Bois-Reymond thinks of these issues?

To answer this, you need to find mathematical counterparts to the words there that are used (implicitly) metaphorically.

[76] Du Bois-Reymond discovered *nerve action potential* and proposed the chemical nature of synaptic transmission.

[77] *Über die Grenzen des Naturerkennens,* 1872 and *Über die Grenzen des Naturerkennens: Die sieben Welträtsel,* 1891.

For instance what is "DETERMINISTIC Universe"?

Most likely, Du Bois-Reymond carries in the back of his mind Laplace's idea of *une intelligence qui, pour un instant donné, connaî trait toutes les forces ...* that models the universe by a system of ordinary differential equations.

This *model* is deterministic due to the unique solvability of such systems under due regularity conditions.

Of course, this is absurd, nobody would literally think that the state (?) of the Universe is given by a set $u(t) = (u_1(t), \dots, u_k(t)) \in U = \{u_1, \dots, u_k\}$ at a moment (?) t by EXACT values of some real parameters u_i and that determinism amounts to some technical uniqueness theorem.

However, one can make sense of it by formulating the question in the following terms.

Selective Instability of Dynamical Systems. Let $F = F_1$ be the "future" map from U to itself, for $F : u(t = 0) \mapsto u(t = 1)$. Given a point u in U define its *instability profile* $IP(u)$ as the set of those u' in the "vicinity" of u for which "essential parameters" of $F(u)$ in U undergo "improbably large" (nearly discontinuous) variations under "small variations" of u in $IP(u)$.

Selective Instability of a state u signifies in this terms some particular mathematical property of the set $IP(u)$ that corresponds to the vague[78] idea of being "structurally interesting".

States u of the Universe that depict *physical* systems $P = P(u)$ are (tacitly) assumed NOT to be *selectively unstable.*[79]

But if there is something *alive* about u, e.g., a protein molecule $M(u)$, a cell $C = C(u)$ or, even better, an "intelligent" humanoid ape $A = A(u)$, then $IP(u)$ does look very interesting. And it may (or may not) be possible to correlate particular features of $IP(u)$ with what $A(u)$ calls "my free will" thus refuting what Du Bois-Reymond says about it.

In order to make the above "rigorous" one needs to specify various terms used: "small variations", "improbably large", etc., which is done by "adjusting logical parameters" with a use of a representative set of examples. Below is one.

Unlocking The Door. Let your universe be reduced to a room with a door and a lock in it, with a key K that may be in any position in this room and a mechanical contraption/apparatus A with say 100 degrees of freedom and such that, according to your record, it is equally likely to occupy all its own states.

If A is "non-intelligent" and, at $t = 0$, the key in a state u is positioned, say in the middle of the room, then the probability of the door being unlocked at the moment $t = 1$, if it was locked at $t = 0$, is virtually zero.

[78]If you are a student of ergo-logic, there is nothing vague for you in the word "interesting".

[79]Yet, some physical systems exhibit some "selectively unstable features", such as localisation/concentration of energy to a few (selective) degrees of freedom.

But if you observe that this probability greatly increases whenever some appendage of A holds K, then, you bet, A is intelligent.[80]

"Nature of mathematics" is absent from Du Bois-Reymond's list. Probably, he did not realize there was "ignorabimus problem" in mathematics, while Hilbert who had thought about it emphatically stated several times that all mathematical problems will be *eventually* solved, and he even suggested a program for *mathematically* proving this assertion.

Where do we stand to-day?

The twentieth century taught us humility. We are not even certain any more what exactly "eventually" means for human civilization. Our time may be short.

And the fascinating depth of the present day *non-understanding* of the physical world – matter, force, energy, motion – was inconceivable to people of the 19th century, who were oblivious of the ideas of relativity and quantum fields and who possessed no knowledge of the large scale dynamics of the observable Universe.

Also, contrary to what Hilbert believed, his naively "optimistic" conjectures on the logical organization of mathematics were mathematically *disproved*.[81]

Mathematics is not a "logical bore" as Hilbert thought, but a miraculous structure that has no right to exist neither intrinsically nor in how it is intertwined with fundamental physics and with human minds.

(Inexplicably, despite the failure of Hilbert's program, the questions we ask in mathematics are getting answered almost instantaneously – often in the course of significantly less than a million of man-hours of people seriously thinking on these problems. But the answers are, sometimes, far from the expected ones.

Probably, it is not hard to design a random device generating mathematical problems that admit simple solutions but these will be not only unfindable but *unfathomable* in realistic time, say in a billion years of intense research by a prosperous Galactic civilisation with a quadrillion of active mathematicians in it. But the human brain is, apparently, unable to formulate such problems at will.[82])

And we believe that the fundamental questions about Nature cannot be formulated but in mathematical terms. The deepest nature of the two arrows

$$\textsc{Space/Time/Matter/Energy} \underset{?_1}{\rightsquigarrow} \textsc{Life/Brain}$$

[80] All this suggests the existence of a rigorous, yet remaining interesting, theory of *selective instability*; possibly, the other of the above seven questions (with possible exception of 1 and 2), if you concentrate hard, may also direct you toward non-trivial mathematics.

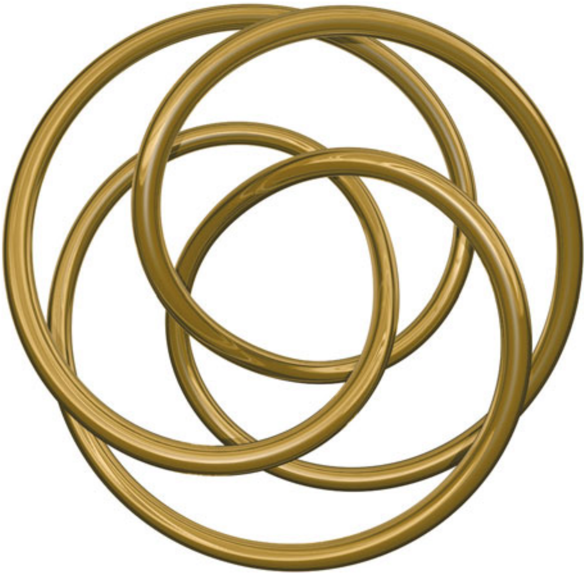
[81] Gödel has shown (1931) that there are unquestionably *true but unprovable statements* in mathematics. More dramatically, Paul Cohen proved (1963) that there are several parallel worlds of mathematics, such that some statements that are *true in one of the worlds may be false in another one.*

[82] Maybe, the P$\neq$NP problem is of this kind.

and

$$\text{Life/Brain} \underset{?_2}{\rightsquigarrow} \text{Mind/Thought}$$

cannot be comprehended but in the ambience of Mathematics.



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The Great Circle of Mysteries is closed with

$$\text{Brain/Mind/Thought} \underset{?_3}{\leftrightsquigarrow} \text{Mathematics}$$

and

$$\text{Mathematics} \overset{?_4}{\leftrightsquigarrow} \text{Space/Time/Matter/Energy}.$$

We know something about $?_4$ – mathematical physicists tell us story after story about it. But no mathematics available to us elucidates the mysteries of $?_1, ?_2$ and $?_3$. The time of this mathematics is yet to come.

Memorandum Ergo

1. Brain, Ergo-Brain, and Mind

The universe is built on a plan the profound symmetry of which is somehow present in the inner structure of our intellect.

PAUL VALERY

Decoding the Mind is impossible without creating a broad (semi)mathematical context allowing one to consistently speak of *mind-like structures*.

But

what kind of mathematics do we need to speak about the Mind?

Should we stick to *mathematics of numbers* – the language physicists speak about their World?

Some think that no radical departure from physics is needed. Frances Crick,[1] for instance, believed – we mentioned this earlier – that much of the mind may be understood in terms of the physiology of the brain:

a person's "mental activities are entirely due to the behavior of nerve cells, glial cells, and the atoms, ions, and molecules that make them up and influence them."[2]

Nobody argues that the sole source of your thoughts *is* your brain – this idea has been around for more than 4000 years.[3] But no matter how much you adorn this idea with persuasive words, everything you say about the arrow

$$[\text{BRAIN}] \rightsquigarrow [\text{mind}]$$

remains metaphoric. No sentence of a kind: "*The mind is*

caused/produced/generated/constructed or *determined/controlled/run*

About two thirds of this "memorandum" is compiled from slightly edited extracts from the two our earlier ergo-articles.

[1]Francis Harry Compton Crick (1916–2004), who greatly contributed to molecular biology, was educated as a physicist.

[2]This is written in Crick's 1994 book *The Astonishing Hypothesis*, where he promotes what he believes is a scientific approach to the problem of consciousness.

[3]A witness to this is *Edwin Smith Surgical Papyrus* (≈ 1500 BCE) – an incomplete copy of a text from the Old Kingdom of Egypt (circa 2686–2181 BCE).

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M. Gromov, *Great Circle of Mysteries*, https://doi.org/10.1007/978-3-319-53049-9_2

by the brain" sheds any light on the nature of this arrow. We cannot but accept[4] what Charles Sherrington grudgingly acknowledged nearly a century ago:[5]

> *"As followers of natural science we know nothing of any relation between thoughts and the brain, except as a gross correlation in time and space."*

But what is wrong with this? What other relations do you want?

Your own brain, or rather, what we call *ergo-brain*, reconstructs the whole world in all its grandeur from space/time correlations between different events.[6] With a little mathematics, we may try something similar as follows.

Different types of brain injury produce different psychological impairments,[7] and experimental neurophysiology (ideally) delivers a correspondence between states of mind and collections of neurons in the brain that are active in the presence of such states. Since the anatomy of the brain is, roughly, the same for all people, this allows an objective comparison of similar collections in different individuals.

For instance, if experiencing a particular color, such as ☻, were universally identifiable by records of excited neurons in the brains of a representative[8] group of people (animals), one would be justified in attributing the "predicate of *existence*" to *the quale* of this color.

More interestingly, the natural combinatorial distance, called *Hamming metric* between different collections of neurons in the brain[9] gives us a way to measure distances between states of mind.

If such a distance/metric were a reality, psychology would be equated with "*geometry of the mind*(*s*)" and albeit no such metric is available with the current state of knowledge, its very idea suggests the possibility of a mathematical approach to the study of the mind.

But, on the other hand, it seems there *cannot be* any "mathematics of the Mind": no matter how much you try you cannot discern anything "mathematical" in what you consciously perceive as "my Mind"; it is too loosely organized with no structurally significant patterns visible in it.

Well . . . , if you watch soap operas on the screen of your computer you do not see much structure either. You have to look somewhere else.

> *It is in the admission of ignorance and the admission of uncertainty that there is a hope for the continuous motion . . . in some direction. . . .*
>
> Richard Feinmann

[4]Crick would disagree.

[5]This quote, which deserves being presented for the second time, is taken from Sherrington's book *Man on his Nature* based on his Gifford Lectures (1937–1938) where the scientist expounds his philosophical ideas on man's place in the universe from the point of view of the natural sciences.

[6]Space-time itself, as it is represented by the ergo-brain, results from such correlations.

[7]This had been already recorded in the Surgical Papyrus.

[8]Very likely, the so-defined quale of WHITE for the Inuit in the arctic regions of Greenland would correspond to GREEN for the Pirahã people of Amazonia.

[9]This distance is defined as the number of neurons that belong *only to one* of the two collections.

Ergo-Brain Conjecture. There exists a certain elaborate mental entity, we call it *ergo-brain*, that mediates between the electrophysiological dynamics of the brain and the thought processes in the conscious mind.

Ergo-brain is responsible for *deep learning* by humans, in particular for learning mother tongues by children and mathematics by future mathematicians.

Little of the ergo-brain is accessible to introspection. Yet, "ergo-patterns" are recognizable in natural languages and within mathematics.

Ergo-Structures/Ergo-Systems Conjecture. There are particular mathematical, essentially combinatorial, structures, call them *ergo-structures*, and a class (mathematical category?) of mathematics objects, called *ergo-systems*, that carry within themselves such structures. Ergo-brain is a particular instance of an ergo-system.

Our ultimate goal is developing the theory of ergo-structures that would bring *mathematical* means for *analysing and synthesising universal learning systems.*[10]

We imagine such a system $\mathcal{LERNER}$ that interacts with *an incoming flow of signals* similarly to a photosynthesizing plant growing in a stream of photons of light or to an amoeba navigating in a sea of chemical nutrients and/or of smaller microbes: $\mathcal{LERNER}$ recognizes and selects what is *interesting* for itself in such a flow and uses it for *building* its own structure.

This analogy is not fully far fetched. There is no *significant* difference between human activities and those by amoebas and even by bacteria, well, ... on the GRAND SCALE. Say, the probability of finding the first million digits of the number $\pi = 3.14159265359\ldots$ "written" at some location of an imaginable UNIVERSE increases by more than a billion-by-billion-by-billion factor if you find a bacterium machine feeding on a source of almost amorphous *free energy* at a point within many thousand light years of this location.

Ergo-logic: this is a particular way of thinking that is needed to approach our conjectures.

Ergo-logic sharply contrasts with everything we take for granted about what we are and what happens in our heads. We reject such ideas as

intelligent – rational – intuitive – important,

as far as ergo-brain and ergo-learning are concerned, and replace them by

interesting – curious – funny – informative.

Albeit counterintuitive, the manifestations of this logic are seen in the depth of mathematics and also in *molecular structures* of living systems uncovered by biologists in the last 50 years.

Also the idea of ergo-brain comes by assessing *limitations of natural selection* in the evolution of Man.

The structural patterns we find in the ergo-brain, although being of evolutionary origin, cannot be accounted for by the naked survival/selection mechanism, but rather by inevitable constrains on possible ergo-system architectures. These

[10]Such a theory may also elucidate the nature of mathematics.

are, essentially, mathematical constraints, and, seemingly paradoxically, they make ergo-brain more likely to be evolutionarily accessible than the "amorphous" human Mind.

And inspired by *the history* of evolution theory, where the insight by Darwin and Wallace was *not so much in biology per se but in realization of potentiality of the exponential function*,[11] we search for the key to the mystery of the Mind *in mathematics rather than in neurophysiology*:

What stands on the way of relating the world of thoughts in your mind to that of neurons in your brain is the disparity of the two structures and incompatibility of the languages describing these structures;[12] amassing data on the Brain will be of little help.[13] The arrow [BRAIN] $\rightsquigarrow$ [MIND] is as defiant of all attempts to contain it in chains of clever words as its younger sister, the arrow

[MATTER/ENERGY] $\rightsquigarrow$ [life].

One *cannot assert* (as Crick would undoubtedly do) that all life processes are *entirely due* to the interactions between atoms, ions, and molecules that make biological entities.

Of course, physicists disagree: Richard Feynman says in *Six Easy Pieces* of his famous *Lectures on Physics* that

> ... *there is nothing that living things do that cannot be understood from the point of view that they are made of atoms acting according to the laws of physics.*

However, the laws of physics are not suspended in a logical vacuum, they are immersed in a mathematical framework. Physics practiced by humans is a "*network of ideas*"[14] within this framework where some "nods" are taken for "laws of physics".

The "spirit of physics" resides in *the combinatorial architecture* of this network that is constrained and directed by many conventions, instructions, assumptions, such as

symmetry, infinitesimal linearity, stability, genericity.

But Life, albeit *constrained* by "physical laws", *excels in violating* "physical conventions and assumptions" – this is what makes Life Life.

---

[11] The enormousness of the exponential growth of unrestricted populations was obvious to mathematically minded people from the time of antiquity. But this might have been a revelation to the biologists of the 19th century who were not well versed in math – Darwin himself, who has a fine intuitive feeling for large numbers, was unable to correctly evaluate the number of descendants of a couple of elephants after 500 years.

[12] This is reminiscent of the *collapse of quantum states arrow* that stands for a (still unavailable) translation of the "quantum language" to the classical one.

[13] This would be like trying to achieve understanding of *proteins* – of their 3D structures and functions in the cell – by accumulating data on chemistry of *polynuclear acids* – DNA – that direct the synthesis of these proteins.

[14] "Idea" may stand for a record of an observation or an experiment as well as a recipe/rule for designing, conducting, and interpreting experiments.

Think, for instance, what happens to a 100 kg, BODY colliding with something tiny, something that weighs less than one billionth of a gram.

Nothing, obviously, but ... let BODY be the body of a *predator*. Let your "something" be a few billion molecules that depart from *the scent glands* in the *body* of a potential *pray* and "collide" with the *olfactory epithelium in the nasal cavity* of BODY.

Would you solely rely on the *Law of conservation of momentum* for predicting the time evolution of the distance between BODY and *body* especially if this second *body* happens to be yours?[15]

The idea of "mathematics of the Mind" is not new. "Algebra of thought" was conceived by Leibniz around 1676.

In 1869, William Jevons[16] built a mechanical *Logic Piano*, that, in his words, represented

> *a mind endowed with powers of thought, but wholly devoid of knowledge.*

In 1887, Charles Peirce[17] was asking how much

> *the business of thinking a machine could possibly be made to perform.*

In 1950, this idea was expounded by Alan Turing in the article *Computing Machinery and Intelligence* where he argues that nothing stands on the way of

BUILDING MACHINES THAT CAN THINK.

But what is *the logical structure* in your mental processes that can be mathematically modelled and implemented on a machine?

The structurally rich *neurophysiology of the brain* is too far removed from what we want to simulate, e.g., the learning process of the mother tongue by a child, but *the flows of your conscious thoughts* are void of interesting structures.

Our suggestion is to switch the focus from *dynamics of the brain* and *logic of thoughts* to

invisible and apparently illogical undercurrents of thoughts

that we collectively call *ergo.*

The core structure of this is determined by the *mathematical* necessity of simplicity and universality, and the shape "ergo" takes in the human mind is influenced by the constraints of neuronal organisation of the brain and by (conjectural) limitations of evolutionary selection.[18]

---

[15] Indian leopards (40–80 kg) and more rarely tigers (150–300 kg) may attack men.

[16] William Stanley Jevons (1835–1882) was an economist and logician. His book *A General Mathematical Theory of Political Economy* (1862) was a start of the mathematical method in economics.

[17] Charles Sanders Peirce (1839–1914), "the father of pragmatism" and the founder of *semiotic*, was an innovator in mathematical logic, philosophy, and statistics.

[18] Think of the shape of a cucumber grown in a bottle.

In the first twenty-one sections of this "memorandum", except maybe for "formalistic" Section 3, we informally discuss what "ergo" might be and in the remaining more technical sections we turn to a concrete analysis of the two basic questions about the Mind, that are:

Is there enough structural universality in the process of "thinking" to allow a mathematical modelling of this process?

What, conceivably, could serve as
THE MATHEMATICS OF THE ERGO-BRAIN?

2. Ergo Project

It is not knowledge, but the act of learning, ...
which grants the greatest enjoyment.

CARL FRIEDRICH GAUSS

The ultimate aim of the *ergo project* is designing a *universal learning program* that upon encountering an *interesting flow of signals*, e.g., representing a natural language, starts *spontaneously interacting* with this flow and will eventually arrive at *understanding* of *the meaning* of messages carried by this flow.

We do know that such programs exist, we carry them in the depths of our MIND, in what we call *ergo-brain*, but we have no inkling of what they are.

Prior to embarking toward the design of such programs we must proceed with

- assessing *flows of signals* commonly encountered in life from a mathematical perspective and formalising what we find *interesting* about them;
- describing, let it be in general, yet, mathematical, terms, what the words *learning, understanding, meaning* signify;
- working out general conceptual guidelines for ergo-learning.

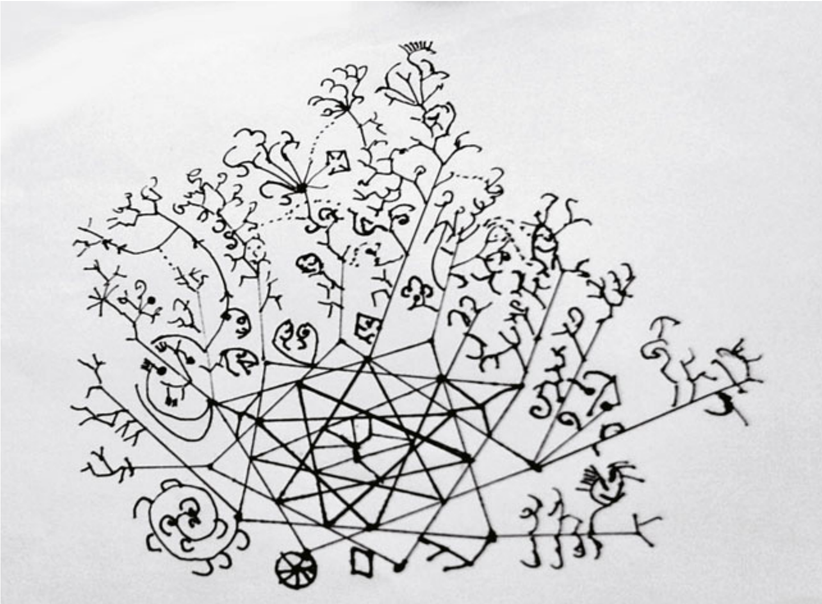
When approaching these issues, one should follow the principles of what we call *ergo logic*, thus *distancing ourselves* from *common sense* ideas about the Human Mind that are dominant in our *ego-mind* and that are pervasive in our (*popular*) *culture.*

Ego-mind is a part of a greater MIND; in fact, it is the part you normally perceive as your mind, but MIND, as we understand it, also contains ergo-brain that, unlike ego-mind, is inaccessible to your mind's eye.

Schematically, MIND is a *finite connected graph* – network of ideas – that is composed of two subgraphs (very roughly) corresponding to the ergo-brain and the ego-mind,

$$\mathrm{MIND} = \mathrm{M}_{\mathrm{ergo}} + \mathrm{M}_{\mathrm{ego}}$$

where M_{ergo} is a kind of a *core* of the MIND, that is a *union of cycles* and M_{ego}, a *periphery*, is a disjoint union of *trees*[19] such that each of these trees meets M_{ergo} at a single vertex – the root of this tree from where it grows.[20]



Ego is *rational*: common sense – the logic of "ego" carries accumulated evolutionary wisdom needed for our personal survival and that of our genes. Common (popular) culture is a kind of collective ego-mind.

Human ergo is *irrational*.[21] It is after beautifully interesting structures in the world, not practically useful ones, it is enthralled by play, art, science. Science and mathematics are at the core of our *collective ergo.*

Common-sense ideas and opinions, unlike the ideas of science, are unquestionably self-evident; you are not suppose to overrun them. For instance,

> *if something heavy falls on you – dodge out of the way as fast as you can: Heavy objects fall faster than the light ones and they hit you harder.*

This is great for your survival. But nothing of this kind – nothing suggested by your common sense – is good as an idea in science.

We do not claim we know which model of the human mind is nearest to the truth, but it must be as dissimilar from what intuition and common sense whisper in your ear as one is capable to imagine.

[19] *Trees* are connected graphs without cycles.
[20] Of course, all finite graphs decompose this way.
[21] Man is least rational of all animals. No matter what a cockroach does, even if it gets killed in the process, its behaviour is 100% rational. You cannot say this about people.

3. Formality and Universality – Meaning, Folding, and Understanding

Sources and channels of flows of signals we perceive may be different – visual, auditory, tactile. But all these enter the brain in the same form:

arrays of "strings" of fluctuating electrochemical currents.

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SIGNALS → BRAIN → MIND → MEANING QUESTION. How much of the *meaning* encoded in such "strings" can be recaptured and *understood* by *formal symbolic* manipulations following some *universal mathematical* rules?

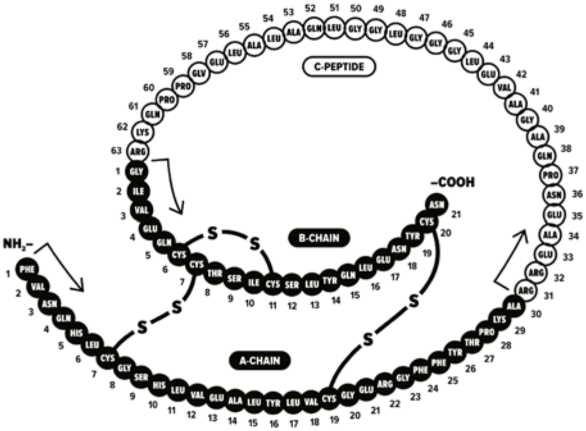
Meaning and *understanding* are elusive concepts – there is no definition of them suitable for approaching this question. But drawing a formal parallel with the central problem in *molecular genetics* is helpful.

GENOTYPE $\rightsquigarrow$ PHENOTYPE QUESTION. How much of the *anatomy, physiology, behaviour* of an organism – bacterium, plant, or animal – can be reconstructed by a *formal* analysis of the *DNA sequence* of the genome of this organism?

Life effectuates the green arrow "$\rightsquigarrow$" by several developmental processes of organisms of which we single out the following two.

⋆ *Protein folding* and ★ *Embryonic development.*

Concerning $\star$, recall, that proteins are synthesised in cells as *polymer chains* from 20 (sometimes 21) molecular units – *common amino acids*.[22] (The smallest amino acid, *glycine*, is composed of 10 atoms and the largest one, *tryptophan*, has 27 atoms.)



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Then such a chain "folds" and thus arrives at a specific (often "potato shaped") *spatial structure* (*conformation*)[23] that determines the *physiological function of the protein*. This function is *reason d'etre* of protein, its *meaning* in the eyes of the cell.

The (stochastic) dynamics of the folding of a loosely positioned flabby string $\mathcal{S}$ of amino acids takes place in a watery environment E, where it is propelled by interactions of $\mathcal{S}$ with randomly moving water molecules[24] and also with other molecules and ions whenever these are present in a significant concentration in E.

But what mainly defines the final compact protein form[25] P is the interactions, kind of (but not quite) *attraction* forces, between constituents of $\mathcal{S}$ when they come in close contact one with another.

Metaphorically, the physical chemistry of the environment *understands* the sequence of $\mathcal{S}$ and extracts its *meaning* by forming P.

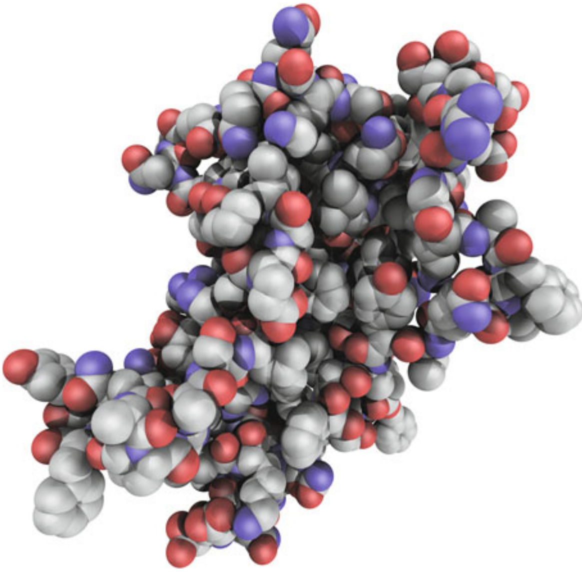
---

[22] The synthesis of proteins effectuates *translation* of what is written in four letters on mRNA's, that are, essentially, replicas of fragments of DNA, to the 20-letter proteins language. This, probably the most complicated chemical process in the Universe, is accomplished by large proteins + RNA aggregates, called *ribosomes*, that are assisted by dozens of other proteins and RNA molecules.

[23] This applies to (many but not all) moderately short chains, say of 50–300 units, where some protein molecules are formed by several chains. For instance the molecule of *haemoglobin* (in the blood of adult humans) incorporates two identical 141-chains and two 146-chains – 574 amino acids altogether.

[24] The (quadratic) average speed of water molecules at the room temperature is about 650 m/s.

[25] This form, unbelievably for a mathematician, is (essentially) unique and it is *determined*, up to a controllable stochastic error, *by the amino acid sequence*, for most proteins in living organisms.



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Then the physiological environment of the cell implements the message encoded in $\mathcal{S}$ by letting P perform the protein P function that the amino acid sequence of P was fashioned for.

Physically, folding can be divided into elementary steps/motions – bending, twisting, and stretching of chemical bonds of an $\mathcal{S}$-molecule. These go in parallel at different locations of $\mathcal{S}$, up to 10^{14} motions per second at each location, and happen simultaneously in many copies of $\mathcal{S}$ in the cell.

But the mechanical speed of the molecular dynamics would not suffice for fast folding of proteins that is observed in the cell – seconds or less, if not for a helping hand of evolutionary biology: *Native* protein sequences are "designed", among other things, for fast folding under physiological conditions.

Computationally, *ab initio* protein structure prediction (granted full knowledge of intermolecular interactions[26]) remains well out of range of the fastest computers.

Experimentally, if – this is a big IF – one succeeds in crystallization of a protein, then its structure can be determined (this is non-trivial) on the basis of *X-ray images of the crystals.*

Bioinformatically, a protein $\mathbf{P}_{\text{new}}$ is studied by comparing its sequence with those of related proteins the structures of which have already been resolved, where the final determination of the structure of $\mathbf{P}_{\text{new}}$ depends on *stereochemistry* of proteins.

[26]The available data on these is far from being complete and accurate.

Logically, the arrow

$$\text{sequence} \mapsto \text{structure}$$

transfers the sequential information in $\mathcal{S}$ to that encoded by the geometry of the *outer surface* ∂P of P.[27] This goes with a significant loss (some say "compression") of information: Different sequences may result in essentially the same protein structures.[28]

Although even the *concepts* of a protein P as "a physical body" is not expressible in the language of $\mathcal{S}$-sequences if P is taken *in isolation*, the space $\mathscr{PB}$ of all such "protein bodies", is describable in this language as

> a *subquotient* (quotient set of a subset) of the set of sequences in 20 letters defined via an *equivalence relation* on the subset of the sequences that properly fold to well shaped proteins, where *two sequences are declared equivalent if the corresponding surfaces ∂P have essentially the same geometries.*

Commentaries

• The conditions "properly-folded" and "well-shaped" may vary and define different classes of sequences, but all of these classes must contain the majority of the native (globular?) proteins.

• A "well-shaped" protein with a "properly-folded" sequence does not *generically* perform what can be called "a function" in any living cell; it is *biologically meaningless*, similar in this respect to a grammatically correct but meaningless (this is a generic property) sentence in a natural language.[29]

On the other hand, among "generic properly-folded" sequences, there are many (say, $> 2^{200} > 10^{60} < 20^{300}$) *non-native* ones; i.e., those that have never been and never will be found in nature, but that, upon folding, would perform certain functions as well as – if not better than – native proteins.

• Let the above "essentially" be understood as "approximately". Then our space of "protein bodies" will depend on a kind of approximation being used and, instead of a single $\mathscr{PB}$, we arrive at an ensemble (*small category*?) $\{\mathscr{PB}_a\}$, parametrised/indexed by possible approximations a, where the "ideal space" $\mathscr{PB}$ emerges as a (*projective*?) limit of these $\mathscr{PB}_a$.

Granted a formal *definition* of "protein folding", one may speak of a formal *resolution* of the *protein folding problem* as a set $\mathscr{R}$ of mathematical rules, that,

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[27] "Geometry" here means that of a "colored" surface ∂P, where "colors" represent the physical/chemical properties, such as *polarity and hydrophobicity*, of the amino acids forming this surface.

[28] It debatable if the mapping *sequence* $\mapsto$ *structure* and/or the corresponding *information 3D-transfer maps* possess some kind of (stochastic?) continuity.

[29] Grammar of folding and semantics of functions of proteins in different cells are closer to each other than grammar and semantics of sentences in different human languages.

using the (sample) set $\mathcal{N}$ of sequences of native proteins that are known (or believed) to properly fold as an input, would come up with (something close to) the ensemble $\{\mathscr{P}\mathscr{B}_a\}$.

Such a set $\mathscr{R}$, even one of a moderate size, may exist. But since $\mathscr{R}$ must depend on physical/chemical specifics of intermolecular interactions and due to the computational complexity inherent in these interactions, such an $\mathscr{R}$ hardly can be found on the basis of *universal mathematical* principles *with no a priori idea* of geometry and physics of folding and/or with no knowledge of the origin of $\mathcal{N}$.

The solution of the protein folding problem by purely formal mathematical reasoning is impossible.

However, certain fragments (mathematically speaking, *subquotients*) of the ensemble of spaces $\{\mathscr{P}\mathscr{B}_a\}$ that represent some "meaningful features" of proteins, e.g., persistence of certain sequential patterns, say, those corresponding to *α-helices*, are easily formally detectable. And a comprehensive mathematical model of proteins open to an input of experimental data may furnish a solution of the folding problem.

The colors of logic change in a variety of ways when we look at the arrow

protein structure $\rightarrow$ *protein function.*

($*$) *Collectivity of Protein Functions.* The function of a given protein P in a cell makes sense only in the ambience of other proteins that interact with P. (This is formally similar to the *function* or *semantic meaning* of a sentence S in a text: Unlike the syntax of S, the meaning cannot be defined out of an extended lexical context.)

Accordingly, it is the space of *(macro)molecular dynamics of the cell* – $\mathscr{M}\mathscr{D}\mathscr{C}$ in the role of $\mathscr{P}\mathscr{B}$ – not the spaces of functions of individual proteins, that is amenable to a formal representation, where $\mathscr{M}\mathscr{D}\mathscr{C}$ must be represented by a subquotient of the set of DNA sequences of (real and imaginary) cell genomes.

($**$) *Predominance of Biology over Physics.* Certain (not all) specific functions of a protein P depend on the "colored shape" of the surface ∂P that is responsible for direct *physical interactions* of P with other molecules, e.g., proteins, that are similar to the attractive interactions between different loci in the molecular chain of P.

But evolutionarily "designed" organisation(s) of cellular functions of proteins in the cell is very much *unlike* what is seen in natural *physical* systems. There is no physical embodiment of this organisation, nothing similar to "protein body" P.

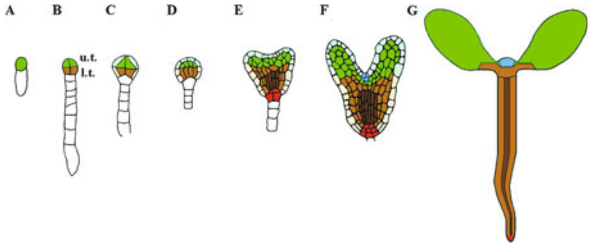
(Some protein functions are reflected, albeit in a limited way, in *protein interaction networks* that record systematic close encounters between proteins molecules in a cell.)

Let us move up the complexity ladder and try to understand in formal terms how the physiological *meaning* encoded in the genome of an organism, that is the *phenotype* of this organism, "unfolds" in the course of embryogenesis.

A possible mathematical format for such understanding can be, as earlier, a representation of the "space" $\mathscr{APO}$ of *anatomies* + *physiologies* of *multicellular organisms* as a subquotient of the space $\mathscr{DNA}$ of (real and imaginary) genome sequences.[30] Achieving this, however, doesn't seem realistic because of the following.

☹ The "space" $\mathscr{APO}$, unlike the (true) protein space $\mathscr{PB}$, is unknown in its entirety and the spaces $\mathscr{APO}_\alpha$ of *accessible attributes* α of organisms, by no means approximate (the largely unknown) $\mathscr{APO}$.[31]

☹☹ One does not know, not even in general terms, what are the rules of *intercellular interactions*, including *cell signalling*, that describe the *biology of morphogenesis.*



Republished with permission of American Society of Plant Biologists, from The Arabidopsis Book 7, Embryogenesis: Pattern Formation from a Single Cell, A. Capron, S. Chatfield, N.Provart, and Th. Berleth, 2009

☹☹☹ No known mathematical model is able to reproduce essential features of morphogenesis, e.g., the resilience of the forms of animals (say, mammals and birds) as they grow from childhood to adulthood.

While the body of a humble bug reads and understands the messages encoded in the bug's genome and implements their *meaning* by building and sustaining itself, our enlightened mind miserably fails to discern the *meaning* in these messages.

And if we cannot match the intelligence of a bug's body do we stand a chance to understand a human brain?

Can we succeed in emulating *the brain's way* of extracting *meaning* from the signals it receives and assigning *meaning* to the signals it generates?

This looks hopeless. The brain's *meaning*, unlike the cell's *meaning* of proteins and of intercellular signals, does not even possess a physical or physiological embodiment – it is a whim of the brain's imagination.

[30]This space $\mathscr{DNA}$ may need to be augmented by additional *epigenetic* information (e.g., by what is molecularly "encoded" in the maternal egg cell), and, possibly, by evolutionary data on organisms and their genomes.

[31]The structure of the ensemble $\{\mathscr{APO}_\alpha\}$, even of the parts of it that have been understood, cannot be represented by anything as logically simple as a mathematical category.

But, paradoxically, this "non-physicality" of *meaning* makes a formal mathematical model of the brain-style understanding of natural signals as well as of unnatural ones – generated by other brains – feasible.

Optimism stems from the apparent *limitations* of the human brain. Signals perceived and emitted by the brain lack fine-tuned specificity of intercellular chemical signals. The brain, unlike the live cell, does not come into direct contact with the environment, it has no internal knowledge of the outside world, nor does it have built in faculties for physiological modelling of the external physical world and/or the world of human relations.[32]

> *The "logical folding" of "sequences" of electrochemical "symbols" received as well as generated by the brain into meaningful shapes – shadows of events and objects in the "real world" – is, by necessity,* (*almost*) *entirely a formal process.*

Mathematically, we think of the *meaning* resulting in this "folding" as an ensemble of subquotients $\mathscr{M}_\alpha$ of the space of (flows of a particular class of) signals $\mathscr{S}$, where *understanding* is a kind of *operational representation* of the arrow that symbolises taking these subquotients,

$$\mathscr{S} \rightsquigarrow \mathscr{M}_\alpha.$$

This is, of course, by no means a proposition of a *solution* to the problem of *meaning* but a suggestion of a possible *language for formulating* the problem. If we accept such a language, then we may search for a solution as a description of a *particular mathematical structure* in the ensemble $\mathscr{M}_\alpha$, and in the ensemble $\mathcal{A}$ of *attributes* α themselves.[33]

The structures of these $\mathcal{A}$ and $\mathscr{M}_\alpha$ as well as of the *arrow of understanding* "$\rightsquigarrow$" depend on a sample $\mathcal{S}$ of signals used for their construction. These structures are elaborately messy – anything but simple and universal.

What is simple and universal, we believe, is the *set of rules of learning* that lead from $\mathcal{S}$ to "$\rightsquigarrow$" and thus, to $\mathscr{M}_\alpha$.

Universality is the most essential property we require from the learning systems/programs which we want to design – these programs must *indiscriminately* apply to *diverse classes* of incoming signals regardless of their "meanings" using the same toolbox of rules for learning languages, chess, mathematics, and tightrope walking.

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[32] The brain is able to map the geometry of the external world to its own internal geometry. Also, the control of motion of (the parts of) the body, e.g., turning the eyeballs and skeletal joints, may be a key factor for the brain's ability to apprehend the *rotational symmetry* of 3-space. But these *non-symbolic representations* hardly lie at the core of human higher cognitive abilities such as language and sequential reasoning.

[33] We say "ensemble" and "attribute" in order not to commit ourselves prematurely to precise mathematical notions, such as *set* for instance.

Without universality there is no chance of a non-cosmetic use of mathematics;[34] and only "clever mathematics" may furnish universality in learning.[35]

But in most of what follows we explain the ideas of ergo-learning with no appeal to "abstract" mathematics.

TERMINOLOGY: *Quotient, Reduction, Categorization, Compression*

Given what mathematicians call a *set* S, that is, a well-defined[36] "collection" of "objects" a *quotient set* $\underline{S}$ of S is obtained by identifying a kind of "gluing" or "binding together" of certain elements in S, where the "rule of gluing", of pairs of members of S, customary written as $s_1 \sim s_2$, is called the *equivalence relation* that defines $\underline{S}$.

(A simple example is where S consists of strings, say of length 50 in 40 symbols: 26 letters, 13 punctuation marks and the "space" symbol, where two strings are declared equivalent if they differ only at the last 5 places.)

Symbolically, the passage from S to $\underline{S}$, called *factorization*, or the *quotient map*, is represented by an arrow $Q : S \to \underline{S}$, also written as $s \mapsto \underline{s} = Q(s)$, where such a Q sends S *onto all* of $\underline{S}$. Pictorially, $\underline{S}$ is a kind of shadow of S defined by the arrow of the rays of light.

(In the above example of strings, the quotient set $\underline{S}$ consists of all strings of length 45 and applying Q amounts to dropping the last 5 letters in the strings.)

Conversely, given an onto map Q from S to another set T, gluing or identification of s_1 and s_2 in S can be expressed by the equation $Q(s_1) = Q(s_2)$ and then T can be regarded as a quotient set of S for the equivalence relation

$$s_1 \underset{Q}{\sim} s_2 \Leftrightarrow Q(s_1) = Q(s_2).$$

A subquotient of a set S is a quotient of a part (subset) P of S. That is, the equivalence relation and the corresponding quotient map are defined only on P.

(For instance, P may be the "set" of *meaningful* English sentences expressed by strings of fifty symbols in the $26 + 13 + 1$ "alphabet", where two such strings are declared *equivalent* if they carry *the same meaning*. Here, the ambiguity of "meaningful", and "same meaning" necessitates considering not individual P and Q but rather a family of maps $Q_{\alpha,\beta}$ defined on subsets P_β in S.)

[34] This is meaningless unless you say what kind of mathematics you have in mind. Mathematical creatures, such as, for example, *Turing machine* and *Pythagorean theorem*, differ one from another as much as a single-stranded RNA virus form a human embryo.

[35] Our objectives are different from those taken by *mathematical psychologists* (e.g., Robert Duncan Luce and James Tarlton Townsend) – *Logical and Mathematical Psychology: Dialectical Interpretation of Their Relations* by Nicolae Mărgineanu, Presa Universitară Clujeană, 1997; *Mathematical Psychology: An Elementary Introduction*, by Clyde H. Coombs, Robyn M. Dawes, and Amos Tversky; Englewood Cliffs, N.J.: Prentice Hall, © 1970. and *Mathematical Psychology: Prospects for the 21st century* by James T. Townsend (2008). Also see http://www.indiana.edu/~psymodel/publications_all.shtml as we are not so much concerned with modelling Human Mind but rather the "invisible" processes that shape the Mind.

[36] This "well-defined" makes it problematic, for instance, to say that *meaningful sentences* in a language constitute a set.

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In physics, factorization maps between *spaces of states*, e.g., the maps obtained by dropping some coordinates describing "states", are called *reductions.*

In linguistics, one says *classification* or *categorisation* and subsets of mutually equivalent elements s in S are called *classes* or *categories.*[37]

In our context, the arrow $S \to \underline{S}$ can be called, albeit only metaphorically, *compression of information* or *suppression of redundancies.*

This "suppression", however, does not mean that redundancies are any kind of nuisance; on the contrary, they are crucial for "logical folding" of natural flows of signals similarly to how protein folding relies on redundancy of information encoded by residue sequences. No meaning can be extracted from a non-redundant flow.[38]

But, unlike how it is with proteins, we have no clear idea of what this "logical folding" is. And it is not just technicalities – what we still miss is the *essence* of mathematics responsible for *understanding* and for *learning to understand meanings* carried by flows of signals.

[37] These have nothing to do with *mathematical categories.*

[38] Thus a logically perfect formalization of a mathematical idea, that redirects mathematical flows from the comfortable channels shared with natural languages to the narrow ditches of thought dug by logicians, impairs understanding of the idea.

4. Universality, Simplicity and Ergo-Brain

Out of chaos God made a world,
and out of high passions comes a people.

BYRON

Our fascination by learning systems comes from what may seem as an almost godlike ability of human (and some animal) infant brains of building a *consistent* model of *external world* from an *apparent* chaos of *flows of electric/chemical signals* that come into it.

Imagine that you see on a computer screen what a baby brain "sees": A *throbbing streaming crowd of electrified shifting points* encoding, in an incomprehensible manner, a certain never seen before, not even imaginable, "reality". Would you reconstruct anything of this "reality"? Would you be able to form such concepts as *shadow*, *roundness*, *squareness*?

Could you extract any *meaning* from a Fourier-like transform of the sound wave the brain auditory system receives?

No. This ability is lost by "mature minds". One cannot even recognize two-dimensional images by looking at graphical representations of the illumination levels, which is a much easier problem. What a baby chimpanzee's brain does is more "abstract" and difficult than the recently found solution of *Fermat's last theorem*.

Yet, we conjecture that an infant's ergo-brain operates according to

a universal set of simple learning rules.

The ergo-brain *extracts structural information* "diluted" in flows of signals following these rules and continuously *rebuilds itself* by incorporating this structure.

(It would be unrealistic to make any conjecture on how such rules could be implemented by the neurophysiology of the human brain, although it seems plausible that they are incorporated into the "architecture of pathways" of signal processing in the brain. But we shall try to guess as much as possible about these rules by looking at the universal learning problem from a mathematical perspective.)

At the moment, one may only speculate in favour of universality by appealing to "evolutionary thrift of Nature" and to "brain plasticity" where, besides propensity for learning languages, convincing evidence for simplicity and universality of the performance of ergo(brain) is the human ability to learn mathematics.

It may strike you as paradoxical that something as *complex* as learning 1000 pages of math and coming to understand the proof of Fermat's last theorem can be effectuated by a *simple* program in your ergo-brain. But a specialized and/or complicated *evolutionarily designed* learning program could hardly do mathematics that is far removed from the mundane activities the brain was meant for.

Ultimately, we want to write down a *short* list of *general* guidelines for "extracting" *mathematical structures* from *general* "flows of signals". And these flows may come in many different flavours – well organized and structured as mathematical deduction processes, or as unorderly as "*a shower of little electrical leaks*" depicted by Charles Sherrington in his description of the brain.

Of course, *nontrivial* structures can be found by a learning system (be it universal or specialized), only in *interesting* flows of signals. For instance, nothing can be extracted from fully random or from constant flows.

But signals that are modulated by *something meaningful* from "the real world" carry within them certain *mathematical structures* that the brain of a human infant can detect and reconstruct.

Universal Patterns in Animal Behaviour

When, as by a miracle, the lovely butterfly
bursts from the chrysalis full-winged and perfect, ...
it has, for the most part, nothing to learn,
because its little life flows from its organisation
like melody from a music box.

Douglas Spalding[39]

Why Universal? It would be unrealistic to expect that evolution had time enough to select for many long sequential learning programs specific to different goals, but "clever" universal programs may accommodate several different situations.

Example: "Hawk/Goose" Effect. A baby animal, as we described this earlier, distinguishes *frequently* observed shapes sliding overhead from those that appear *rarely.*

The former eventually stops soliciting HIDE! responses, but every unusual kind of a shape, e.g., that of a hawk, makes an animal run for cover.

Such universal programs develop in the environment of evolutionary older "*general ideas*" that are kinds of *tags*, such as dangerous/harmless, edible/useless, etc., selectively associated in an animal/human brain with a variety of *particular* "real somethings" in the world.

NO universality, however, in biology is mathematically perfect – "laws of biology" are no brothers of laws of physics.

The most general law of Life from a biologist's point of view is *the genetic code* – a *specific* correspondence between two sets accidentally frozen in time, one is comprised of 61 out of 64 triplets of four bases – adenine, thymine, cytosine,

[39] Douglas Spalding (1840(41?)–1877), the founder of *ethology*, was arguably, along with Gregor Mendel, the most original biology (and psychology) thinker of the 19th century. He discovered *imprinting* in baby animals (popularised by Konrad Lorentz in the 1930s) and he began the study of anti-predator reactions. Unlike Darwin and Freud, he had not exposed his ideas to laymen and his name remained unknown to the general public.

and guanine – in DNA –, and the second one is the set of 20 basic amino acids in proteins.[40]

$$\{AAA, AAT, \ldots, TTT, AAC, \ldots, GGG\}$$

$\rightsquigarrow$ {❀, ✲, ❊, ✾, ❁, ❉, ✻, ✹,✸, ✤,✺,✷, ✥,✳,✶, ❄,❅,❁, ✤,✿,✖}.

But the primal universality in biology for an ergo-minded mathematician is seen in

⋆ *one-dimensionality* of polynucleotides and polypeptides,

⋆⋆ *digital* nature of the genetic code,

⋆⋆⋆ *information 3D-transfer principle* implemented by *folding of biological heteropolymers.*

Evolution, Universal Grammar and Chomskyan Theory

> *... ideas which consist of "symbolic images". The first step to thinking is a painted vision of these inner pictures ... which are produced by an "instinct to imagining" and ... re-produced by different individuals independently ...* — Wolfgang Pauli

According to Chomsky, Lenneberg, and their followers, *poverty of stimulus* (i.e., limited data) would prevent children, who have amazing innate ability for language acquisition, from learning mother tongues as quickly as they do unless they have *universal grammar* "unscripted" in their LAD – *Language Acquisition Device* – a "language organ" in the brain or rather a module of the human mind that emerged in the course of human *evolution* as a result of some peculiar *mutation.*

Linguists usually do not bother furnishing any specific genetic or neurophysiological data on this "mutation" but rather operate with concepts of "evolution" and "mutation" *metaphorically.*

It is more likely, contrary to what Chomsky insists upon, that

> *language...* [is] *not a wholesale innovation, but ... a ... reconfiguration of ancestral systems.*[41]

And from our ergo perspective, "universal deep learning mechanisms" are not limited to language, but also carry a variety of other functions, such as

- learning to read and to write by ~95% of (non-dyslexic) people on Earth;
- learning mathematics by mathematically inclined students that make 10%–30% of all people;[42]
- learning to play chess by particularly gifted children (this is said about Morphy, Capablanca, Tal, Waitzkin) by observing adults play.

[40] Three nucleotide triplets: TAA, TAG, and TGA are *stop codons* that do not correspond to amino acids. Yet TGA may represent the 21st amino acid: *selenocysteine.*

[41] *The eloquent ape: Genes, brains and the evolution of language*, Fisher & Marcus [15].

[42] It may be close to 30%(100%?) at ages 3–5; eventually it declines, probably, below 1%, partly under the pressure (unconsciously) exerted by "mathematically dyslectic" parents and teachers.

The presence of these abilities in human populations forcefully denies naive arguments of Darwinian[43] adaptive evolution.

Technical (Im)Practicality of Universality. Multi-purpose gadgets are not among the Greatest Engineering Achievements of the Twentieth Century: *flying submarines*, if they were a success, then only in James Bond movies.[44] On the other hand, 20th century machine computation has converged to universality; basic machine learning will, most probably, follow this path in the 21st century.

5. Freedom, Curiosity, Interesting Signals, and Goal Free Learning

The essence of mathematics lies in its freedom.

GEORG CANTOR

These words by Georg Cantor equally apply to learning in place of mathematics. Universality necessitates the *non-pragmatic* character of learning. Indeed, *formulating* each utilitarian goal is *specific* for this goal – there is no universal structure on the "set of goals". Thus,

the essential mechanism of learning is goal-free
and independent of external reinforcement,[45]

where the primary example of free learning is the first language acquisition.

The ability of native learning systems to function with

no purpose, no instruction, no reinforcement

is no more paradoxical than, say, a mechanical system moving in the absence of a force.

External constraints and forces change the behaviour of such systems, but *inertia* remains *the source* of motion. (The use of metaphors in science leads to confusion. The force of gravity is what *makes* things fall but it can hardly be called the *source* of motion of the Earth around the Sun.)

Closer to home, think of your digestive system. The biochemistry of metabolic networks in the cells in your body needs no teacher instruction, albeit hunger initiates the digestive process.

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[43] More often than not, "Darwinian" is used synonymously to "truly scientific". But our "Darwinian" refers to how evolution was understood prior to the genetic revolution in the last decades of the 20th century.

[44] There are sea birds, e.g., *pelagic cormorants* and *common murres*, who are (reasonably) good flyers and who also can dive, some up to more than 50 (150?) m. The technology for building comparably universal/adaptable machines may be waiting ahead of us.

[45] The feeling of pain when you fall down or bump into something may be helpful in learning to run – this is debatable; but contrary to what a behavioristically minded educator would think, *reward/punishment reinforcement* does not channel the learning process by *reinforcing* it, but rather by *curtailing and constraining* it. Compare [33], [38].

Similarly, you may start learning to play chess or to walk a tightrope in order to impress your peers, but the learning program(s) in your (ergo)brain carries no trace of this purpose.

Ergo-Systems. These are universal learning systems that we want to design. They also must be self-propelled learners that learn spontaneously with no need for instructions and reinforcement. (Strictly speaking, our concept of ergo-system is broader, in particular it does not exclude native ergo-brain learners.)

Curiosity as Intrinsic Motivation. The idea of what we call *ergo-systems* is close to what was earlier proposed by Schmidthuber [29] and by Oudeyer, Kaplan, and Hafner [25] under the name of *Intrinsically Motivated Curiosity Driven Robots.*

This "motivation" is implemented by a class of *predictor programs*, that depend on a parameter B that is coupled with (e.g., by being a function of) the behaviour of robots.

These programs Pred = Pred(H, B) "predict" in a certain specified way incoming signals on the basis of the history H, but robots (are also programmed to) optimize (in a specific formally defined way) the quality of this prediction by varying B.

Thus, "freedom" for an ergo-brain is not just a possibility to generate any kind of signals it "wants", but rather to have "interesting" environmental responses to these signals.

For instance, a bug crawling on an *infinite* leaf has *zero* freedom: No matter where it goes it learns nothing new. But an accessible edge of the leaf, adds to the bug's "freedom".

Similarly,[46] an ergo-brain comes to "understand" the world by "trying to maximize" its "predictive power" but what exactly it predicts at every stage depends on what structure has been already built. The "architecture of understanding" in the human mind is built from "bricks of predictability" that come in all shapes and colors. (This may be hard to reconcile with Rene Thom's "*Prédire n'est pas Expliquer*".)

In order to maximise anything, one needs some freedom of choice, e.g., your eye needs a possibility to run along lines/pages or, in a chess game, you can choose from a certain repertoire of moves.

When this repertoire becomes constrained, the ergo-brain feels *bored* and *frustrated.* This happens to you when a pedantic lecturer curbs your curiosity by displaying slides on the screen line by line, preventing you from seeing the whole page.

And a most dramatic instance of being prevented from learning is described by Helen Keller:[47]

[46] The distribution density of a bug's positions crawling on a leaf is similar to that of your eye scanning this very leaf.

[47] Helene Keller, who lost her sight and hearing at the age of 18 months, was not exposed to *tactile sign language* until nearly age seven.

> *Once I knew only darkness and stillness ... my life was without past or future ... but a little word from the fingers of another fell into my hand that clutched at emptiness, and my heart leaped to the rapture of living.*

The idea of "*interesting*" – that is, the feature of a structure that excites "curiosity" of a learner – can be best grasped by looking at the extreme instances of *uninteresting* flows of signals – the constant ones:

●●●

○○○

There is (almost) nothing to predict here, nothing to learn, there is no substance in this flow for building your internal ergo-structure. (If you were deprived of freedom to learn by being confined to an infinite flat plane with no single distinguished feature on it, you would soon be mentally dead; boredom cripples and kills – literally, not metaphorically.)

And random *stochastically constant* sequences do not look significantly more interesting.

○●○●●○○●●○●○○○●●○○●●●○●●●○○●●○○○○●●●○●○○●●○●○●○●●●●○●○

This appears "non-interesting" because one loses control over incoming signals, but there is much to learn from the following string that makes your curious ergo much happier.

Our ergo idea of "interesting" is suspended in balance between *maximal novelty* of what comes and *being in control* of what happens.

(Pure randomness looks boringly uneventful to your eye but your *vestibular* and the *proprioceptive/somatosensory systems*[48] would enjoy propelling your body through a rugged terrain with occasional random jumps from one rocky stone to another making the trip enjoyably dangerous.[49])

6. Information, Prediction, and a Bug on the Leaf

The Optimal Prediction idea of Schmidthuber–Oudeyer–Kaplan–Hafne is central in our thinking on ergo-systems but we emphasize "structure" instead of "behaviour", with *degree of predictability* being seen as a part of the structure of flows of signals within and without an ergo-system.

This "degree" is defined as a function in three (groups of) variables:

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[48] These sensory systems tell you what are the current (absolute and relative) positions, velocities, and accelerations of your body and of its parts, with most accelerations being perceived via stresses in your skeletal muscles.

[49] *Irrationality* is a hallmark of humanity. Only exceptionally, grown-up *non-human* animals are able to derive pleasure from doing something that carries no survival/reproduction value tag attached to it.

the learner system LEARNER *and two fragments, say* $\overleftarrow{past}$ *and* $\overrightarrow{future}$ *in the flow of signals, where* LEARNER *predicts "something" from* $\overrightarrow{future}$ *on the basis of its knowledge of* $\overleftarrow{past}$*.*

This "something" refers to the result of some reduction – a kind of simplification procedure applied to $\overrightarrow{future}$ *where such a reduction may be suggested by* LEARNER itself or by another ergo-system, e.g., by a human ergo-brain.

An instance of this is predicting a *class of a* [*word*] in a [*text*] on the basis of several preceding words or classes of such words. Such a class may be either syntactic, such as *part of speech*: verb noun . . . , or semantic, e.g., referring to

vision, hearing, motion, animal, inanimate object,

or something else.

And "degree of predictability" of a class of [*word*] derived from correlations of this class with words that *follow as well as precede* [*word*] is also structurally informative.

(The *proper direction*, that is "follow" versus "precede" relation, is *not intrinsic* for (a record of) a flow of speech: it is non-obvious if strings have to be read left to right or right to left in an unknown language.

But, possibly, the direction can be reconstructed via some *universal* feature of the "predictability (information) profile" of such a flow *common to all* languages[50], similarly (but not quite) to how the arrow of time is derived from the evolution of macroscopic observables of large physical ensembles.)

© Kathy Pilato,
http://katpilato.blogspot.ch/2011/02/lady-of-different-sort.html

Let us apply the prediction idea to a bug crawling on a leaf or of your eye inspecting a *green* spot on a *brown* background.

[50]Phonetics of recorded speech suggests an easy solution but it would be more interesting to do it with deeper levels of the linguistic structures. In English, for instance, the correlations of *short* words with their neighbours are stronger for the neighbouring words that *follow short-words* rather than precede them; but this may not be so in other languages.

We assume (being unjust to bugs) that all the bug can perceive in its environment are two "letters" G and B – the colors (textures if you wish) of its positions on the leaf, where the bug has no idea of color (or texture) but it can distinguish green G-locations from the brown ones, B.

The four "words" our bug (eye) creates/observes on contemplating the meaning of its two consecutive positions are GG, GB, BG, BB

But can the bug tell GB from BG?
Is GG "more similar" to GB than to BB?

An algebraically minded bug will translate these questions to the language of *transformations* of the "words in colors":

1. *Switching the colors*: GG $\leftrightarrow$ BB, GB $\leftrightarrow$ BG.
2. *Interchanging the orders of the letters*: GB $\leftrightarrow$ BG with no action on GG and BB.[51]

These transformations do not essentially change the "meaning" of the words: a green square on a brown background is identical, for most purposes, to a brown square on a green background. (Is this "equality" recognized by animals?)

Alphabet of Bug's Moves. Besides perceiving/distinguishing colors, our bug (as well as our eye) has a certain repertoire of moves but it knows as little about them as it knows about colors. Metaphorically, a bug moves by pressing certain "buttons" and then records the colors of the locations these moves bring it to.

The bug does not know its own position on the leaf and the *equality of two moves* – pressing the *same button* – translates in the bug's mind to the constancy of its direction and speed of motion.

(The eye, unlike the bug, "knows" its position s and, in order to repeat a move, it needs to "forget" s. Besides, the eye has several independent arrays of buttons corresponding to different modes of eye movements, some of which are stochastic.)

Amazingly, albeit obviously, this is exactly what is needed for reconstruction of the *affine geometry* of the Euclidean plane that tells you which triples of points • • • in the plane are positioned on a straight line and, moreover, when one of the points, say •, is positioned exactly *halfway* from • to •.

And if the bug can "count" the numbers of repetitions of identical moves, it can evaluate distances and, thus, reconstruct the full *Euclidean* (*metric*) *structure* of the ambient space, that is the 2-plane in the present case.

Which buttons has the bug to press in order to efficiently explore the leaf and learn something about its meaning – the shape of the leaf?

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[51] There is yet another transformation: changing the color of the first letter:
GG $\leftrightarrow$ BG, BG $\leftrightarrow$ GG. This, together with the positional transformations generates a *non-commutative group* with 8 elements in it, called the *wreath product* $\mathbb{Z}_2 \wr \mathbb{Z}_2$; the role this group plays in the life of insects remains obscure.

The bug feels good at the beginning being able to predict that the color usually does not change as the bug makes small moves. (The eye, unlike the bug, can make fast large moves.) But it becomes bored at this repetitiveness of signals, until it hits upon the edge of the leaf. The bug becomes amazed at the unexpected change of colors and it will try to press the buttons that keep it at the edge.

(Real bugs, as everybody has had a chance to observe, spend a disproportionally long time at the edges of leaves. The same applies to human eyes.)

In order to keep at the edge, the bug (this is more realistic in the case of the eye) needs to remember its several earlier moves/buttons. If those kept it at the edge in the past, then repeating them is the best bet to work so in future. (This does work if the edge is sufficiently smooth to be close to straight on the bug's scale.)

Thus, the bug learns the art of the navigation along the edge, where it enjoys twice the predictive power of what it had inside or outside the leaf: The bug knows which color it will see if it pushes the "left" or the "right" buttons assuming such buttons are available to the bug. (The correspondence "left"$\leftrightarrow$"right" adds yet another involution to the bug's world symmetry group.)

Amazingly (accidentally?), this tiny gain in predictive power, which makes the edge interesting for the bug, goes along with a tremendous information compression: The information a priori needed to encode a leaf is proportional to its area, say $A \cdot N^2$ bits on the N^2-pixel screen (where A is the relative number of the pixels inside the leaf) while the edge of the leaf, a curve of length l, can be encoded by $\mathrm{const} \cdot l \cdot N \cdot \log N$ bits (and less if the edge is sufficiently smooth). Unsurprisingly, edge detection is built into our visual system.

(The distribution of colors near the edge has a greater entropy than inside or outside the leaf but this is not the only thing that guides bugs. For example, one can have a distribution of color spots with essentially constant entropy across the edge of the leaf but where some pattern of this distribution changes at the edge, which may be hard to describe in terms of the entropy.)

Eventually the bug becomes bored traveling along the edge, but then it comes across something new and interesting again, the tip of the leaf or the T-shaped junction at the stem of the leaf. It stays there longer playing with suitable buttons and remembering which sequences of pressing them were most interesting.

When the bug starts traveling again, possibly on another leaf, it would try doing what was bringing ut to interesting places before and, upon hitting such a place, it will experience the "déjà vu" signal – yet another letter/word in the bug's language.

We have emphasized the similarities between eye and bug movements but there are (at least) two essential differences.

1. The eye moves much faster than the bug does on the neurological time scale.
2. The eye can repeat each (relatively large) "press the button" move only a couple of times within its visual field.

3. Besides "repeat", there is another *distinguished* move available to the eye, namely *reverse*.[52] Apparently, approximate back and forth movements of the eye appear disproportionally often, especially when comparing similar images.

But the problems faced by our bug are harder than evaluating a metric in a *given* space.

Imagine that you are such a bug at a keyboard of buttons about which you know nothing at all. When you press a button, either nothing happens – the color does not change – or there is a blip indicating the change of color.

Can you match these buttons with moves on the plane and the blips with crossing the boundaries of monochromatic domains?
What is the fastest strategy of pressing buttons for reconstruction of the shape of a domain?

The answer depends, of course, on the available moves and the shapes of the domains: You need a rich (but not confusingly rich) repertoire of moves and the domains must not be too wild.

What you have to do is to create a language, with the letters being your buttons – blips, such that the geometric properties of (domains in) the plane would be expressible in this language of sequences of pressing the buttons marked by blips. If in the course of your experiments with pressing the buttons, you observe that these properties (encoded by your language) are satisfied with significant (overwhelming?) probability, you know you got it right.

But what the bug has to do is even more difficult, since, apparently, there is *no a priori idea of spatial geometry* in the bug's brain.[53] Bug's geometry is the grammar of the "button language".[54]

Because of this, the bug's brain (and an ergo-brain in general) cannot use a strategy tailored for a particular case, but must follow *universal* rules, as the real bugs, we believe, do. Success depends on the relative simplicity/universality of plane geometry, more specifically on the group(s) of symmetries of the plane. (This symmetry is broken by "colored" domains in it, and, amusingly, *breaking* the symmetry makes it perceptible to an "observer" – the bug or the eye at the keyboard.)

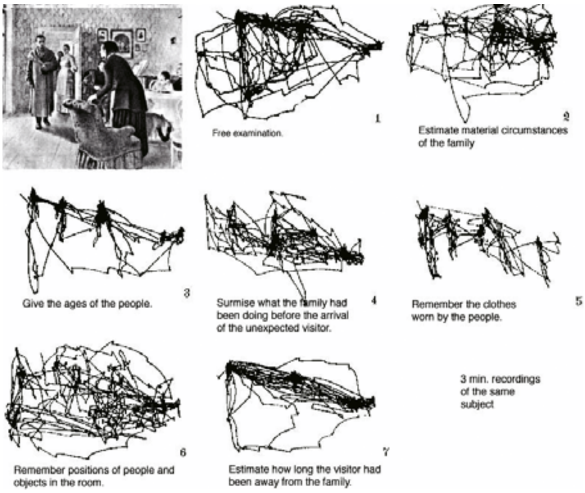
And the bug is able to make an adequate picture of the world, because, incredulously, *universality is universal*:

[52] Geometric incarnations of "reverse" are more amusing than the affine spaces associated with "repeat". These are *Riemannian symmetric spaces*.
[53] Some animals, e.g., mice, have *map making programs* in their brains.
[54] This is similar in spirit but dissimilar in every single detail from axiomatic representations of geometries by mathematicians.

The mathematical universality of a bug's strategies matches
the universal mathematical properties of the world.

This universality shapes the mechanisms of your mental processes as much as those of a bug. Your eye spends more time focused at the *edges* of images as much as a bug that crawls along the edges of leaves, and you pay most attention to the ends of words and it usually doesn't mcuh mttaer in waht oredr the ltteers in a wrod are.

> *Eye movements reflect the human thought processes; so the observer's thought may be followed to some extent from records of eye movements... The observer's attention is frequently drawn to elements which do not give important information but which, in his opinion, may do so. Often an observer will focus his attention on elements that are unusual in the particular circumstances, unfamiliar, incomprehensible, and so on.*
>
> Alfred Yarbus.



Eye Movements and Vision, Eye Movements During Perception of Complex Objects, 1967, page 174, Alfred L. Yarbus (Springer-Verlag US 1967)

7. Stones and Goals

Reach high, for stars lie hidden in you.

Rabindranath Tagore

Some maintain that mastering accurate throwing, a uniquely human[55] capacity, could have been, conceivably, a key factor in early hominid brain evolution.[56] According to *the unitary hypothesis*, the same neural circuitry may be responsible for other sequential motor activities, including those involved in speech production and language [5], [39].

We cannot judge the neurophysiological plausibility of this conjecture[57] but there is a chasm of differences between learning to throw and learning to speak as far as mathematics of the two learnings is concerned.

Achieving precision throwing is a *single layer problem*. Any conceivable algorithm for it, no matter how naive – that may not even need the knowledge of the laws of mechanics – is going to work. And nowadays, it is no big deal to build a mechanical contraption that will beat any human in a throwing contest many times over, the best of Palaeolithic hunters included.

It is more difficult, but probably feasible, to design a similar program that would imitate whatever controls your tongue and larynx for proper production of sounds. But language is a multilayer structure, wheels within wheels. No one is close to designing "a speaking algorithm" that would come close to a silliest of human conversation.[58]

The unitary hypothesis, regardless whether it is right or wrong, is hardly (?) relevant to our "ergo" but looking at throwing from a position of a goal free learner is instructive.

For a thrower, the most important is his/her *aim*, that must be achieved with a correct *initial condition* – the velocity vector of a stone – that then will follow the trajectory toward a desired target. You may (and you better do) fully forget the laws of Newtonian mechanics for this purpose.

But from a physicist's point of view, it is the *second law* + *the force field* (gravitation and air resistance) that determine the motion – the initial condition is a secondary matter and the destination point is even less so.

A mathematician goes a step further away from the ancient hunter and emphasizes the general idea of time-dependent processes being described/modelled by *differential equations.*

[55] Elephants may be better than humans at precision throwing.

[56] 500 000-year-old hafted stone projectile points, 4–9 cm long, were found in the deposits at Kathu Pan in South Africa, http://www.newscientist.com/article/dn22508-first-stonetipped-spear-thrown-earlier-than-thought.html

[57] There is a parallelism between spontaneous drives to learn to speak and to learn to walk/run/jump by children, but precision throwing is not in the same basket.

[58] An exception is what a patient expects to hear during a seance of psychoanalysis: ELIZA – a program written by Joseph Weizenbaum in the mid 1960s – successfully imitates a psychotherapist.

We – physicists and mathematicians with all our science would not stand a chance against *Homo heidelbergensis*[59] in a spear throwing contest; however, for instance, we, at least some of us, shall do better in mathematically designing *gravity-assist trajectories* from Earth to other Solar System bodies.

But from the position of Homo heidelbergensis[60] it would be unreasonable, not to say plain stupid, to aim at an inedible target.

Well, let us make it clear, *goal-free learning* is far from being "plain and reasonable" but it rather follows a mathematical physicist in his view on mechanical motion: There is nothing special, nothing *intrinsically* interesting, *neither in the hunter's aim* no matter how hungry he/she is, *nor in the initial condition*, although much skill is needed to achieve it. But the *transformation*

$$\textit{initial position} \mapsto \textit{aim}$$

that incorporates the laws of motion expressed by differential equations, is regarded as something *universal* and the most essential from our point of view.

There are many possible aims and initial conditions but not so many fundamental laws L and of transformations *initial position* $\mapsto$ *aim* associated to them.[61] This what makes these laws so precious in our eyes.

Similarly, one may think of learning as a transformation of an *initial input* and/or of a *learning instruction* to the *final aim* of learning.

Here we are even in a poorer position than the ancient hunter: We have hardly an inkling of what the corresponding "transformation by learning" does as it brings you from the initial input/instruction to your aim:

What is the "space" where all this happens?

What is the structure of "the trajectory" for *initial input* $\underset{?}{\rightsquigarrow}$ *aim of learning*?

And, unlike a teaching instructor, we are not concerned with *observable* inputs and aims but with mathematical models of *invisible* intrinsic structures of transformations *inputs* $\to$ *aims* that are built according to "universal laws of learning".

It is not that we deny the importance of goals, instructions, and external stimuli for learning, but we relegate them to secondary roles in the "transformation formula" that is responsible for the arrow *inputs* $\to$ *aims*. We try to understand learning processes regardless of their specific aims, or, rather, we want to see general *aim generating mechanisms* within the "universal laws" of learning.

---

[59] *Homo heidelbergensis*, a probable ancestor of Homo sapiens as well as of Neanderthals and of Denisovans, lived in Africa, Europe, and western Asia between 1 000 000 and 200 000 years ago.

[60] This position is articulated by Lev Tolstoy in an essay on science where he speaks for *a plain and reasonable man*. But of course not all (if any) of Heidelberg men were plain and reasonable. Those who were have returned to the trees.

[61] This stands in sharp contradiction with *Cantor's theorem*: There are more *logically conceivable* functions $f : x \mapsto y = f(x)$ than arguments x. But logic should not be taken literally when it comes to "real life mathematics".

8. Ego, Ergo, Emotions, and Ergo-Moods

One may understand the cosmos, but never the ego;
the self is more distant than any star.

GILBERT K. CHESTERTON

Our main premise is that learning mechanisms in humans (and some animals) are *universal, logically simple, and goal-free*. An organized totality of these mechanisms is what we call *ergo-brain* – the essential, albeit nearly invisible, "part" of human mind – an elaborate mental machine that serves as an *interface* between the neurophysiological brain and the (*ego*) mind.

Metaphorically, this "invisible" is brought into focus by rewriting the Cartesian

I THINK therefore **I** AM

as

cogito ERGO sum.

"I think" and "*I am*" are what we call *ego-concepts* – structurally shallow products of *common sense*. But ERGO – a mental transformation of the seemingly *chaotic flow* of electric/chemical signals the brain receives into a coherent picture of *a world* that defines your personal idea of existence – has a beautifully organized *mathematical structure*.

Our MIND, as depicted on p. 79, contains two quite different separate entities, that we called *ego mind* and *ergo-brain*.

Ego-mind is what you see as your personality. It includes all that you perceive as your conscious self – all your thoughts, feelings and passions, with subconscious as a byproduct of this *ego*.

The rational and intelligent "ego", shaped by the *evolutionary selection* that acted on tens of millions of generations of our animal forebears, and that serves your survival and reproduction needs, also carries imprints of *the popular culture* of the social group an individual belongs to.

Ego-processes are observed in the behaviour of humans and animals and some are perceived by retrospection.

Ego-mind is "real", large, and *structurally shallow*. Most (all) of what we know of ego-mind is expressible in *the common sense* language that reflects the logic of ego-mind. This language is adapted to our social interactions; also it suffices for expressing ideas in *the theory of mind* of *folk psychology*.

Ego-mind is responsible for WHYs about your thoughts; if you want to understand HOWs you must turn to the *ergo-brain*.

Ergo-brain, logically, mediates between electrochemical dynamics of neuronal networks in the brain and what we perceive as our "thinking".

Ergo-brain is something abstract and barely existing from ego's point of view. Ultimately, ergo-brain is describable in the language of what we call (mathematical universal learning) *ergo-systems*, but it is hard to say at the present point what

ergo-brain truly is, since almost all of it is invisible to the conscious (ego) mind. (An instance of such an "invisible" is the mechanism of *conditional reflexes* that is conventionally regarded as belonging with the brain rather than with the mind.)

Ergo-brain, unlike ego-mind, is a structural entity, which underlies deeper mental processes in humans and higher animals; these are not accessible either to retrospection or to observations of behaviour of people and/or animals. This makes the ergo-brain difficult (but not impossible) for an experimental psychologist to study. (Folk psychology, psychoanalysis and the like are as unsuitable for looking into the depths of the mind as astrology is for the study of the synthesis of heavy atomic nuclei in supernovae.)

The ergo-brain and the ego-mind are autonomous entities. In young children, human and animals, the two, probably, are not much separated; a presence of *ergo in the mind* is visible in how children think about play.

As the ego-mind ("personality", in the ego-language) develops it becomes protected from the ergo-brain by a kind of a wall.[62] This makes most of ergo-brain's activity invisible.

In grown ups, ergo, albeit reluctantly, may comply with demands by ego:

"*Concentrate and solve this damn problem! – I need a promotion.*"

But the two can hardly tolerate each other.

Human ergo has a seriousness of a child at play. As a child, it does not dutifully follow your instructions and does not get willingly engaged into solving your problems. This irritates ego. From the ego perspective, what ergo does, e.g., composing utterly useless chess problems, appears plain stupid and meaningless.

Reciprocatory, utilitarian ego's activity, e.g., laboriously filing in tax return forms, is dead boring for ergo.

Certain aspects of ergo may be seen experimentally, e.g., by following *saccadic eye movements*, but a direct access to ergo-processes is limited.[63]

But there are properties of the working ergo in our brain/mind that are, however, apparent.

For example, the *maximal number* $N_\circ$ of concepts our ergo-brain can manipulate *without structurally organizing them* ("chunking" in the parlance of psychologists) equals three or four.[64] This is seen on the conscious level but such a bound is likely to apply to all signal processing by the ergo-brain.

For instance, this $N_\circ$ for (the rules of) chess is between three and four: The three unorganized concepts are those of "rook", "bishop" and "knight", with a weak structure distinguishing king/queen.

[62] Dramatic effects of accidentally breaking this wall are described in [10], [2], [26], [34], [35].

[63] This is similar to how it is with cellular/molecular structures and functions, where the "ergo of the cell", one might say, is the machinery controlled by the *housekeeping genes that is not directly involved in any kind of production by the cell.*

[64] Some people claim their $N_\circ$ is as large as (Miller's) "magical seven" but this seems unlikely from our mathematical perspective; some psychologists also find the number four more realistic.

Similar constraints are present in the structures of natural language where they bound the number of times operations allowed by a generative grammar may be implemented in a single sentence.[65]

Animal (including human) emotional responses to external stimuli are rather straightforward with no structurally elaborate ergo mediating between neuronal and endocrine systems.

Think of emotions as colors or typefaces – a few of dozen different kinds of them, which the brain may choose for writing a particular message, such as

run! run! RUN! RUN!

Ergo-moods also come in different colors:

curious, interested, amused, amazed, *perplexed, bored,*

serve as indicators as well as dynamic components, of the activity of the ergo-brain. These indicators tell us how far our ergo-brain is from animal rationality.

Our visual system is *amused* by optical illusions, *amazed* by tricks of magicians, *fascinated* by performances of gymnasts.

Our auditory system is *enchanted* by music.

Our olfactory system is attracted by exotic perfumes.

Our gustatory system is hungry for strange and often dangerously bitter foods.

Our motor/somatosensory system plays with our bodies making us dance, walk on our hands, perform giant swings on the high bar, juggle several unhandy objects in the air, climb deadly rocks risking our lives, play tennis, etc.

Ergo-moods, being independent of the pragmatic content of the signals received by the ergo-brain, serve as universal signatures/observables of ergo-states.

These moods are apparent as reactions to *external* signals by the ergo-brain; we conjecture that similar signatures mark and guide *the internal ergo-processes* as well.

9. Common Sense, Ergo-Ideas and Ergo-Logic

Einstein, when he says that

common sense is the collection of prejudices acquired by age eighteen

does not try to be intentionally paradoxical. There is a long list of human conceptual advances based on *non-trivial* refutations of the *old way which is also the common-sense way.*[66] The first entry on this list – *heliocentrism* – was envisioned

[65] An often repeated statement that "one *can* potentially produce an *infinite* number of sentences in any language" is, to put it politely, a logical misdemeanour.

The only meaningful concept of "infinite" belongs with mathematics and there is no room for the concept of "can" within mathematics proper. (Hiding behind "potentially" or appealing to such definition as "a language is a set of strings . . . " does not help.)

[66] This is Lev Tolstoy's idea of how *a plain, reasonable working man* should think.

by Philolaus, albeit not quite as we see it today, twenty four centuries ago. The age of enlightenment was marked by the counterintuitive idea of Galileo's *inertia*, and the 20th century contributed *quantum physics* – *absurd from the point of view of common sense* – in Richard Feynman's words. (Amusingly, Einstein sided with common sense on the issue of *quantum*.)

It is hard to withstand the command of common sense with its incessant buzz in your head and murmur in your guts. The patulous tree of ego-wisdom in your mind, as much as your emotional self seeping with "noble feelings", is the net result of millions years of evolutionary pruning – "*the fittest survives*". And the final form of your ego-self was moulded by equally brutal cultural pressure of the last few thousand years. Unsurprisingly, a pragmatically teleological ego-centred mode of thinking that was installed by evolution into our conscious mind along with the cauldron of high passions seems to us intuitively natural and logically inescapable. But this was selected by Nature not at all for a structural modelling of the world including the mind itself.

Ergo-ideas flow from a different source and their orientation is anything but pragmatic. Ergo, unlike ego, was not specifically targeted either by evolutionary selection nor by the pressure of a popular culture – it was adopted by evolution out of sheer logical necessity. (This is similar to the *one-dimensionality* of DNA molecules that is not a result of any kind of selection as it has never been in competition with, say, two-dimensionality.) Non-accidentally, ergo is often in discord with our intuition and with the dominant cultural traditions of our social environment.

> *The first principle is that you must not fool yourself –*
> *and you are the easiest person to fool.*
>
> Richard Feynman

The self-gratifying ego-vocabulary of

> *intuitive, intelligent, rational, serious, objective,*
> *important, productive, efficient, successful, useful*

will lead you astray in any attempt at a rational description of processes of learning; these words may be used only metaphorically. We *cannot*, as Lavoisier says,

> *improve a science without improving the language or nomenclature that belongs to it.*

The intuitive common sense concept of *human intelligence* – an idea insulated in the multilayered cocoon of *teleology* – purpose, function, usefulness, survival – is a persistent human illusion. If we want to understand the *structural essence* of the mind, we need to break out of this cocoon, wake up from this illusion, and pursue a different path of thought.

It is hard, even for a mathematician, to accept that your conscious mind, including basic (but not all) mathematical/logical intuition, is run by a blind evolutionary program resulting from "ego-conditioning" of your animal/human ancestor's minds by a million years of "selection by survival" and admit that

mathematics is the only valid alternative to common sense.

Yet, we do not fully banish common sense but rather limit its use to concepts and ideas *within* mathematics. To keep on the right track we use a semi-mathematical reasoning – we call it *ergo-logic* – something we need to build along the way. We use, as a guide, the following

Ergo list of Ideas:

interesting, meaningful, informative, funny, beautiful,
curious, amusing, amazing, surprising,
confusing, perplexing, predictable, nonsensical, boring.

These concepts, are neither "objective" nor "serious" in the eyes of the ego mind, but they are *universal*. By contrast, such concept as "useful", for instance, depends on what, *specifically*, "useful" refers to.

Hopefully ergo-logic and ergo-ideas will direct us toward developing ergo-programs that would model learning processes in children's minds. After all, these minds can hardly be called serious, rational, or objective.

It is difficult to bend your ego-mind to the ergo-way of thinking. This, probably, is why we have been so unsuccessful in resolving the mystery of the Mind.

Chimpanzee Model

We cannot learn much about ergo by a study of animal behaviour,[67] but our egos are similar to those of animals. This is seen in the following experiment performed by Sarah Boysen more than 20 years ago.

X (Sarah) and Y (Sheba) were chimpanzees who learned the concepts "more than" and "less than" and who adored gumdrops, the more the better.

While Y watched, X was asked to point to one of the two plates on the table: A "large" one, with many gumdrops and a "small" one, with few of them. Whichever plate X pointed to was given to Y.

Try after try, X pointed to the "large" plate and received only a few gumdrops. Apparently X realized it was behaving stupidly but could not override the "grab what you can" drive.

Then gumdrops were replaced by plastic chips. Now X invariably pointed to the "small" plate and thus received more gumdrops than Y.[68]

[67] There are exceptions. Orangutans, for instance, have a propensity for $3D$ *topology*. They may enjoy playing with knots as much as human mathematicians do.

[68] Abstract "more" and "less" are not ingenious of ergo-brain as we shall argue later on. Chimpanzees' "more"/"less" for food (depending on the intensity of the smell?) and for non-edible items might be located in mutually disconnected parts of their brains/minds.

We invite the reader to find yet another interpretation of this experiment besides this and the obvious one.

10. Ergo in the Minds

Those who dance are often thought mad
by those who hear no music.

TAO TE CHING

The most dramatic evidence for the existence of an unbelievably powerful survival indifferent mental machinery in our heads comes from the rare cases where *the ergo insulating wall* has "*leakages*".

The legacy of evolution keeps "ergo-power" in our minds contained by an *ego-insulating wall*: A hunter-gatherer whose ergo-brain had overrun his/her pragmatic ego-mind did not survive long enough to pass on his/her genes.

But if in olden times, people with such "leakages" in their ergo-brains had no chance for "survival", in today's civilized societies they may live; they shine like "mental supernovas" unless their fire has been stifled by educational institutions.

Srinivasa Ramanujan (1887–1920) was the brightest such supernova in the Universe of Mathematics; only accidentally, due to the intervention of Godfrey Harold Hardy, did he get a chance to become visible.

When he was 16, Ramanujan read a book by G.S. Carr, "A Synopsis of Elementary Results in Pure and Applied Mathematics", that collected 5000 theorems and formulas. Then in the course of his short life, Ramanujan wrote down about 4000 new formulas, where one of the first was

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}}=3.$$

During his life, Ramanujan recorded his discoveries in four notebooks. The fourth notebook – a bunch of loose pages – the so-called "lost notebook", with about 650 of Ramanujan's formulas, most of them new, was rediscovered in 1976 by George Andrews in the library at Trinity College.[69]

Judging phenomena of "Ramanujans" and "Mozarts" *statistically insignificant*[70] is like signing off *explosions of supernovae* to *mere accidents*, just because only a dozen supernovae were recorded in our galaxy with billions of stars (none since 1680).

The hidden mental power of everybody's (ergo)brain, not only of Ramanujan's brain, must be orders of magnitude greater than what is available to the ego-mind, since *rare* mental abilities could not have been evolutionarily selected

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[69] George Andrews and Bruce Berndt collaborated on publishing four volumes, appearing in 2005, 2009, 2012, 2013, of the proofs of Ramanujan's formulas included in the lost notebook. http://www.math.uiuc.edu/~berndt/lostnotebookhistory.pdf.

[70] Ergo-logic, unlike insurance companies, assigns *significant* weights to *miraculously improbable* events.

for and structurally complex functional features (be they anatomical or mental) cannot come by accident.[71]

What kind of mathematical structure could adequately describe the "mysterious something" in the human brain/mind that caused the transformation from the flow of written symbols from Carr's book to the formulas written by Ramanujan?

Unless we develop a fair idea of what such a structure can be, we would not accept any speculation either on the nature of mathematics or of the human mind, be it suggested by psychologists or by mathematicians.

Further evidence in favour of ergo-brain – a universal mathematically elaborate machine hidden in *everybody's* head that is responsible for non-pragmatic mechanism(s) of learning can be seen in the following.

1. *Spontaneous learning mother tongues by children.*

Albeit human speech depends on our inborn ability to distinguish and to articulate a vast variety of phonemes, the structural core of learning mother Language goes according to some *universal rules* that are not bound to a particular physical medium supporting a "linguistic flow". Learning languages and writing poetry by deaf-blind people is evidence for this.

2. *Learning to read and to write.*

This, unlike learning to speak, has no evolutionary history behind it.

3. *Mastering bipedal locomotion.*

One is still short of designing bipedal robots that would walk, run, and jump in a heterogeneous environment.

4. *Human fascination by sophisticated body movements: dance, acrobatics, juggling.*

5. *Playful behaviour of some animal, e.g., human, infants during the periods of their lives when the responsibility for their survival resides in the paws of their parents.*

6. *Attraction to useless* (*from a survival perspective*) *activities by humans, such as climbing high mountains and playing chess.*

Albeit rarely, adult animals, e.g., dolphins, engage in similarly useless playful actions.

7. *Creating and communicating mathematics.*

Probably, several hundred, if not thousands or even millions, people on Earth have a mental potential for understanding *Fermat's last theorem,*[72] by reading a thousand-page *written proof* of it.

The following example demonstrates human ergo in all its illogical beauty.

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[71] The development of the brain is a random process, where only its general outline is genetically programmed. Rare fluctuations of some average "connectedness numbers" can be further amplified by "Hebbian synaptic learning". To properly account for this one has to argue in terms not of individual ergo-brains but of *(stochastic) moduli spaces of ergo-brains.*

[72] No integers $x > 0, y > 0, z > 0$ and $n > 2$ satisfy $x^n + y^n = z^n$.

A 4–5 year-old child who sees somebody balancing a stick on the tip of a finger, would try to imitate this; eventually, without any help or approval by adults, he/she is likely to master the trick.[73]

© Jenn Huls / Shutterstock.com

What is the mathematics behind this?

A naive/trivial solution would be reformulating the problem in terms of classical mechanics and control theory. The balancing problem is easily solvable in these terms but this solution has several shortcomings:

- It does not apply where the external forces are unknown.
- It does not scale up: No such robot came anywhere close to a healthy human in its agility.
- It suggests no universality link between balancing sticks and Ramanujan's $\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$.

A better (ergo-style) solution of the balance problem with a single degree of freedom – the inclination angle α, may be obtained by following $grad_v(T)$ for $T = T(\alpha, \alpha', v)$ being the empirical "falling time" where α' denotes the angular velocity and v is the control parameter – the (horizontal) velocity of the support.

But even this distracts you from the key issue:

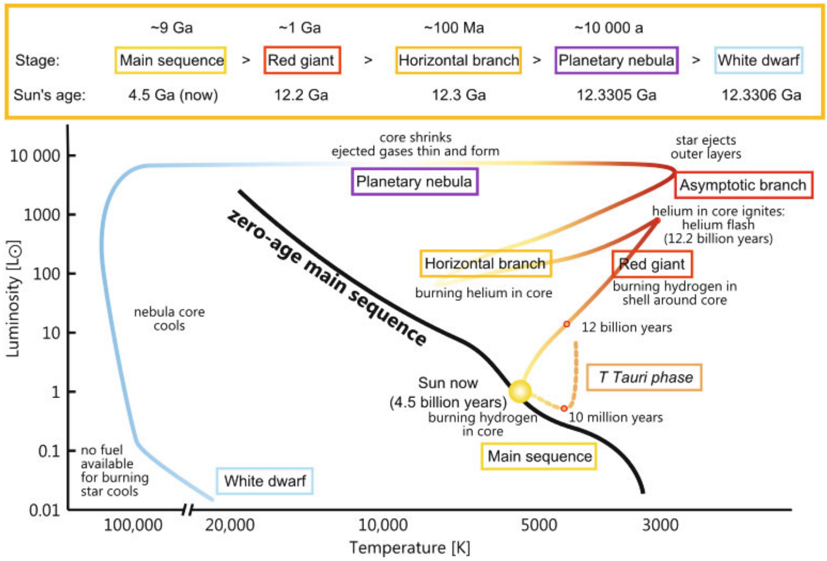
What on Earth drives children to try to perform such tricks?
What drives Ramanujans to invent impossible formulas?

(Younger children at play enjoy putting pencils vertically on their non-sharp ends on the table. And if captured and caged by an extraterrestrial being unable to write Ramanujan-style formulas, you would have to prove your "non-animal level of mentality" by putting a stick vertically in the *centre* of your cage.)

[73] Possibly, let it be rarely, a baby ape may also try to do it, but a *reasonable* human or non-human grown-up animal would have none of this nonsense.

About Stars

There are between 100 billion and 1000 billion stars in our galaxy with less than 10 thousand visible by the naked eye from Earth. One estimates that there are 2–4 supernovae explosions per century in our galaxy.[74]



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The majority of stars do not become supernovae. For instance, the life expectancy of Sun-like stars is about 10 *billion* years and their ends are *relatively* peaceful.

Stars that are ten times bigger than the Sun shine 10^4 times brighter and live only $\approx$ 10 *million* $(=(10 \cdot 10^{10})/10^4)$ years. They end up exploding as supernovae. Also some stars in binary systems turn to supernovae by accreting matter from their companions.

During several weeks a supernova radiates with an intensity of 1 billion–100 billion Suns.

[74]The brightest supernova in the 19th century sky of science, as it is seen from the position of the 21st century, was the 1866 article *Versuche über Pflanzen-Hybriden* by Gregor Mendel, who derived the *existence of genes* – atoms of heredity by a statistical analysis of the results of his experiments with pea plants. The world remained blind to the light of this star for more than 30 years.

11. Language and Languages

... may well have arisen as a concomitant of structural properties of the brain.

NOAM CHOMSKY

Most of what we know about the structure of the human ergo-brain is what we see through the window of human language. This "window" is brain's own invention, kind of a *half-step* toward *meaning* of signals it receives and it produces. Formally, in the sense of Section 3, this is a decomposition of the (subquotient) signals/meaning arrow $\mathscr{S} \rightsquigarrow \mathscr{M}_\alpha$ as

$$\mathscr{S} \rightsquigarrow \mathscr{LANG} \rightsquigarrow \mathscr{M}_\alpha.$$

But what is LANGUAGE in simple terms? Is it *conversing, writing, reading?*

What are essential *mathematical* structures characteristic for languages?

What would make signals coming from space classified as "*language*" rather than "*music*" or *a record of an elaborate chess-like game*?

We forfeit any idea of definition[75] but rather sketch in a few words a picture of languages of the world.

One counts ≈ 7000 currently spoken[76] (or having recently become extinct[77]) languages where *distinct languages* (rather than distinct dialects) are defined by linguists as *clusters* of vertices in the graph $\mathcal{L}_{Earth}$ where the nodes of $\mathcal{L}_{Earth}$ represent "something" (parlances) spoken by small communities of people and where the (weighted) edges of $\mathcal{L}_{Earth}$ express (degrees of) *mutual intelligibility* of these "somethings".[78]

Languages with traceable common origins are organized into ≈ 150 *language families.*

Whistled languages. There are several dozen regions in the world where the speakers of native languages can also communicate by whistling, thus transmitting messages over distances of several distances.

Silbo, based on Spanish, is practiced by 20 000 inhabitants on the island of La Gomera in the Canary Islands.

Silbo replaces Spanish phonemes with whistling sounds distinguished by pitch and continuity. (Two whistles replace the five Spanish vowels, and four whistles used for consonants.)

---

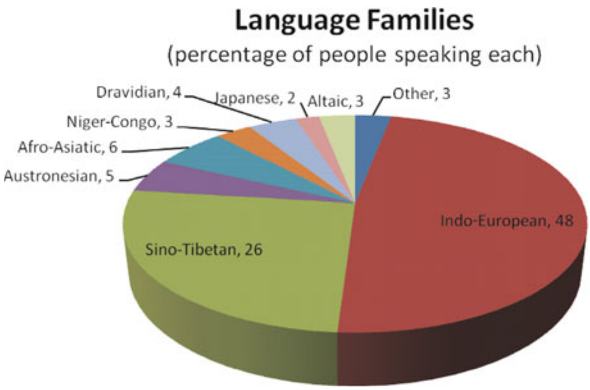
[75] A definition of *language* with a reference to human activity is as helpful for us as defining *plants* by their cooking recipes for a molecular biologist.

And defining *language* as a set of strings of symbols is no better than defining *plant* as a set of atoms.

[76] There are data on writing systems for ≈ 3500 languages (not all of them are widely used) and ≈ 700 languages are known to be unwritten.

[77] There are ≈ 1000 languages each spoken by $\lessapprox$1000 people with about 20–30 languages disappearing every year.

[78] Such an $\mathcal{L}_{Earth}$, of course, is only a dream of a linguist; besides, identifying languages as clusters depends on a *clusterization algorithm* in use.



MRI monitoring shows that whistling sounds are processed in the same localities of the brain as Spanish sentences.

Mazatecan languages, spoken by 200 000 people in southern Mexico, are *tonal*; thus, well adapted to whistling. Whistled communication is used predominately by men but understood also by women in Mazatecan speaking communities.

Pirahã – the language of 200–300 Pirahã people, living in the Amazon rainforest in Brazil, can be whistled, hummed, or encoded in music.

The Pirahã language, unrelated to any other living language, has been studied by Keren and Dan Everett who lived with Pirahã people for nearly 10 years.

According to the Everetts there are no specific names for colors, no plural, and no concept of *number* in Pirahã, possibly not even for "*two*". Many other features of Pirahã, such as sentences not being produced according to Chomskyan-style *transformational grammar,* also remain controversial.

Pidgin and Creole Languages. Pidgins that serve for communication between people, e.g., traders, having no common language, are built from words and other units, of several other languages.

Children who learn a pidgin as their first language create in the process learning a creole language with as elaborate a grammatical structure as those of natural languages and that are missing from pidgins.

Sign languages. There are more than 100 different sign languages in deaf communities in the world. They are mainly independent of spoken languages; their grammars have little (if any) resemblance to that of spoken languages in the same areas. For instance, *British Sign Language* (more than 100 000 users) and *American Sign Language* (more than 300 000 users) are mutually unintelligible.

Sign languages are structurally in the same league as spoken ones, but they have a high *non-sequential* component: Many "phonemes" are produced simultaneously by combining shapes, orientations and movements of the hands, arms, and

body, as well as facial expressions. This makes developing writing forms of sign languages quite difficult; and most sign languages have no written counterparts.[79]

Learning and developing sign languages follow ergo-routes similar to those of spoken languages and a well-minded ergo-blind teacher intervention only serves to block the learning – even more so the language creation – process.

ISN. A unique (?) instance of the emergence of a new language is provided by *Nicaraguan Sign Language* (ISN) that was developed by about 400 deaf children in Nicaragua in the1980s, after attempts to teach children "sign Spanish" had failed, and children became linguistically disconnected from their teachers.

This is considered by some as the strongest evidence for *innate human language capacity.*

Language Acquisition by Deaf-blind Children

> *Deafness is a much worse misfortune* [than blindness]. *For it means the loss of the most vital stimulus – the sound of the voice that brings language, sets thoughts astir and keeps us in the intellectual company of man.* Helen Keller

The bulk of information we receive enters the brain through the eyes and the ears[80] (with more than 50% of the human cortex dedicated to vision), while the external world reconstructed by the brain of a deaf-blind, the world that is defined by what and who he/she touches, must be dissimilar to the world of those who see and hear.

However, given an opportunity, deaf-blind children master languages up to the point of writing poetry. This is amazing.

> *Life is either a great adventure or nothing.* Helen Keller

The great adventure of learning by Helen Keller (1880–1968), who was left blind and deaf at the age of 18 months, commenced when she was six under the guidance of 21 year old Anne Sullivan.[81]

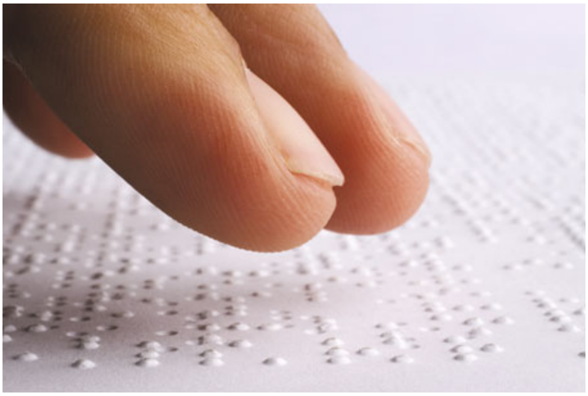
After an initial painful failure to understand the meaning of tactile signs, a dramatic breakthrough came when Keller connected the feeling of cool water running on her hand with the Braille sign for water.

---

[79] Probably, there is an (ergo)algorithm encoding flows of signs by sequences of "phonemes"; thus, delivering *phonetic* representations of sign languages, such that children would be able to learn languages by listening and reproducing such "flows of phonemes"; but hardly an experiment is possible. Compare [18].

[80] Babies and young children are hungry for *tactile information* received from the lips, the tongue, and the hands.

[81] Anne Sullivan (1866–1936) – a brilliant educator, dubbed "miracle worker" by Mark Twain – was herself visually impaired. Born in a poor illiterate family of Irish immigrants she contracted trachoma at the age 7 and nearly fully lost her sight. Anne's education began in 1880, first by learning to read and write and to use the manual Braille alphabet. At that time, she had undergone several eye operations, which improved her sight. Soon after her graduation in 1886 she became the tutor of blind-deaf Helen.



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Supported by Anne's creative teaching adapted to her needs, Helen had learned in a few months 600 words in Braille and the multiplication table. Eventually, Keller had mastered touch-lip reading, typing, and finger-spelling. Later on she learned to speak.

In the course of her life Helen wrote a dozen books as well as a multitude of essays: on faith, on blindness prevention, birth control, the rise of fascism in Europe, and atomic energy. Also she gave many public speeches campaigning for women's suffrage, labor rights, and social equality.

Poetry by Deaf-Blind People

The following gives an idea of how deaf-blind people perceive the world.

. . . .

In the realms of wonderment where I dwell
I explore life with my hands;
I recognize, and am happy;
My fingers are ever athirst for the earth,
And drink up its wonders with delight, . . .

From *A Chant of Darkness* by Helen Keller[82]

. . . .

My hands are . . .
My Ears, My Eyes, My Voice . . .
My Heart.
They express my desires, my needs
They are the light
that guides me through the darkness

. . . .

[82] http://www.deafblind.com/hkchant.html

With my hands I sing
Sing loud enough for the deaf to hear
Sing bright enough for the blind to see . . .

From *My Hands* by Amanda Stine[83]

The free use of ordinary language by deaf-blind people, whose internal model of the external world is quite different from the rest of us, points toward a significant independence of language from non-linguistic stimuli in accordance with ideas of Chomsky.

Language may occupy a tiny part of the brain compared to that committed to vision but, for a human being, as Ludwig Wittgenstein says,

The limits of my language mean the limits of my world.

And the inexplicable ability of children to learn and to use language is most astounding in deaf blind people. But only two lines out of 800 pages in the Handbook of Linguistics[84] are dedicated to this miraculous phenomenon:

Even a child like Helen Keller, who has lost both hearing and sight, can still acquire language through symbols expressed in touch and motion.

In the words of Helen Keller,

The only thing worse than being blind is having sight but no vision.

12. Meaning of Meaning

Everything we call meaningful is made of things that
cannot be regarded as meaningful.
. . . "meaning" is . . . a word which we must learn to use correctly.

Niels Bohr, misquoted[85]

Meanings of words are determined to a large extent
by their distributional patterns.

Zelig Harris

The idea of "*meaning*" advocated by Harris is quite different from the common usage of the word "*meaning*" that invariably refers to "the real world" with "meaningful" being almost synonymous to what is advantageous for preservation and propagation of (observable features encoded by) your genes. (The speakers of the word are usually blissfully unaware of this and they become unhappy if you suggest such an interpretation of the meanings of their actions.)

The former is a *structural meaning* the full extent of which may be discerned only in the dynamics of the learning processes in humans, but the latter, the concept *pragmatic meaning*, is shared by all living organisms, at least by all animals

[83] http://www.deafblind.com/myhands.html
[84] Blackwell Publishing (2007), p. 466.
[85] In the original, one has "real" instead of "meaningful" and "reality" instead of "meaning".

from insects on.[86] This idea of meaning – the commandment to survive – was firmly installed in our brain hardware by evolutionary selection several hundred million years before anything resembling humans came into existence.

A possible way to look beyond a survival-oriented mode of thinking is to turn your mind toward something like chess, something that does not (contrary to what Freudians say) carry a significantly pronounced imprint of the evolutionary success of your forefathers.

But even if you manage to switch your mind from ego- to ergo-mode, you may remain skeptical about (ergo)chess telling you something nontrivial about learning languages and understanding their meanings.

Superficially (this is similar but different to what was suggested by Wittgenstein), one may approach a dialog in a natural language as a chess-like game that suggests an idea of (ergo) meaning: The *meaning* of an utterance UTT is derived similarly to that of the meaning of a position POS in chess. The latter is determined by the combinatorial arrangement of POS within the ergo-structure $\mathcal{CHESS}_{\text{ergo}}$ of "all" ergo-interesting chess positions/games and the former is similarly determined by its location in the architecture of $\mathcal{TONGUE}_{\text{ergo}}$ of a language.

More generally, we want to entertain the following idea that is an elaboration on the "formal discussion" in Section 3.

> *The meanings assigned by ergo-structures* (*e.g., by our ergo-brains*) *to signals are* ***entirely*** *established by patterns of combinatorial arrangements and of statistical distributions of "units of signals", be they words, tunes, shapes, or other kinds of "units".*
>
> *Understanding is a* ***structurally organized*** *ensemble of these patterns in a human/animal ergo-brain or in a more general ergo-system.*

But even leaving aside the lack of precision in all these "pattern", "arrangement", etc., one may put forward several objections to this idea.

The most obvious one is that words, and signals in general, are "just names" for objects in the "real world"; the "true meaning" resides in this world. But from the brain perspective, the only "reality" is the interaction and/or communication of the brain with incoming flows of signals. The "real word" is an abstraction, a model invented by the brain, a conjectural "external invisible something" that is responsible for these flows. Only this "brain's reality" and its meaning may admit a mathematical description and be eventually tested on a computer.[87]

(There are many different answers to the questions "What is meaning?" and "What is understanding?" offered by linguists, psychologists, and philosophers.[88]

[86]Semiotically minded *vervet monkeys* would not hesitate to say that the meaning of the *word-signs* of their language resides in (dangerous) object-events as these come to their fields of vision: a leopard, an eagle, a python, a baboon.

[87]We do not want to break free from the *real world*, but from the hypnosis of the words EXISTENCE/NON-EXISTENCE coming along with it.

[88]References can be found on the corresponding pages of Wikipedia.

We, on the other hand, do not suggest such an answer, since we judge our understanding of the relevant ergo-structures as immature. The expression "*structurally organized ensemble*" is not intended as a definition, but rather as an indication of a possible language where the concept of *understanding* can be productively discussed.)

Another objection may be that learning chess and understanding its meaning, unlike learning native languages by children, depends on specific verbal instructions by a teacher.

However, certain children, albeit rarely – Paul Morphy, Jose Raul Capablanca, Mikhail Tal, and Joshua Waitzkin – as we said earlier, learn chess by observing how adults play. And as for supernovas, it would be foolish to reject this evidence as "statistically insignificant".

More serious problems that are harder to dismiss are the following.

(∘) The structures $\mathcal{TONGUE}_{\text{ergo}}$ of natural languages are *qualitatively* different from $\mathcal{CHESS}_{\text{ergo}}$ in several respects.

Unlike how it is with chess, the rules of languages are non-deterministic, they are not explicitly given to us, and many of them remain unknown. Languages are bent under the load of (ego)pragmatics and distorted by how their syntactic tree-like structures are packed into one-dimensional strings.

Self and Time. The most interesting feature of natural languages is the *self-referentiality* of their (ergo)syntax (e.g., expressed by *pronouns* and/or by certain *subordinate clauses*) that allows languages to *meaningfully* "speak" about themselves.

This is present in most condensed form in *anaphoras* such as in

X thinks he is a good chess player,

and related features common to all human languages are seen in *deixis*, such as in

but I am afraid you may be disappointed by the naivety of his moves,

along with various forms of *grammatical aspects* linked to the *idea of time*.

(It is hard to say how much of *time in the mind* is necessitated by the time dynamics of the neurobrain, what had been installed by the evolution and what comes with flows of incoming signals. And it is unclear if *time* is an essential structural component of the ergo-brain and if it should be a necessary ingredient of universal learning programs.)

None of these have counterparts in chess[89] or in any other non-linguistic structure, e.g., in music. Yet, self-referentiality is seen in mathematics on its borders with a natural language, e.g., when it is communicated from mind to mind and in its logical foundations such as *Gödel's incompleteness theorem*.

[89]Does the "meaning" of the following sentence reside in the game being played or in the conjunction of syntactic self-referentiality loops in there?

I thought I understood why X's white knight was placed on square a1 but his next move caught me by surprise.

($\circ\circ$) The internal combinatorics of $\mathcal{TONGUE}_{\text{ergo}}$ may be insufficient for the *full* reconstruction of the structure of the corresponding language.

For example, linguistic signals a child receives are normally accompanied, not necessarily synchronously, by what come via all his/her sensory systems, mainly *visual* and/or *somatosensory* signals – feelings of touch, heat, pain, sense of the position of the body parts, as well as *olfactory* and *gustatory* perceived signals.

The full structure of $\mathcal{TONGUE}_{\text{ergo}}$ and/or the meaning of an individual word may depend on (ergo)combinatorics of $\mathcal{VISION}_{\text{ergo}}$ coupled with $\mathcal{TONGUE}_{\text{ergo}}$, not on $\mathcal{TONGUE}_{\text{ergo}}$ alone.

$\mathcal{VISION}_{\text{ergo}}$ is vast and voluminous – more than half of the primate (including human) cortex is dedicated to vision, but the depth of the structure of "visual" within $\mathcal{TONGUE}_{\text{ergo}}$ seems limited and cannot by itself support linguistic structures. This is in concordance with the ability of deaf blind people to learn to speak by essentially relying on their *tactile* sensory system; that is, their feeling of touch.[90]

The role of *proprioception* (your body/muscle sense) and the *motor control system* in learning (and understanding?) language is more substantial than that of vision, since production of speech is set in motion by firing motor neurons that activate muscles involved in speech production – *laryngeal muscles, tongue muscles* and hordes of other muscles (hand/arms muscles of mute people); thus, an essential part of human linguistic memory is the memory of sequential organization of these firings.

(Proprioception, unlike vision, hearing, and olfaction, has no independent structural existence outside your body; also it is almost 100% interactive – you do not much feel your muscles unless you start using them. The internal structure of proprioception is quite sophisticated, but, probably, it is by no means "discretized/digitalized" being far remote from what we see in language. It is hard to evaluate how much of language may exist independently of $\mathcal{PROPRIOCEPTION}_{\text{ergo}}$, that may include $\mathcal{TACTILE}_{\text{ergo}}$, coupled with the motor control system, since a significant dysfunction of these systems at an early age makes one unable to communicate.)

The above notwithstanding, (ergo)programs (as we see them) for learning chess and a language, and accordingly, the corresponding ideas of *meaning* and *understanding*, have much in common.

To imagine what kinds of programs these may be, think of an *ergo-entity*, call it $\mathcal{EE}$, from another Universe, to whom you want to communicate the idea/meaning of chess and with whom you want to play a game.

A preliminary step may be deciding whether $\mathcal{EE}$ is a *thinking* entity; this may be easy if $\mathcal{EE}$ possesses an ergo-brain similar to ours, which is likely if ergo is universal.

[90] There are most intriguing differences in the first language acquisition by sighted and blind children, see [3]. Probably, comparable peculiarities would be present in the language acquisition by a sighted child born in a society of blind people who were unaware of their blindness.

For example, let $\mathcal{EE}$ have the mentality of a six-year-old Cro-Magnon child, where this "child" is separated from you by a wall and where the only means of communication between the two of you is by tapping on this wall.

Could you decide if the taps that come to your ears are produced by a possessor of an ergo-brain – more versatile than yours if you are significantly older than six – or from a woodpecker?

If you also happen to be six years old, the two of you will develop a common tap language-game and enjoy *meaningfully* communicating by it, but possessors of two mature human minds separated by a wall will do no better than two adult woodpeckers.

To be a good teacher of chess (or of anything else for this matter), you put yourself into $\mathcal{EE}$'s shoes and think of what and how you could learn from (static) records of games and how much a benevolent and dynamic chess teacher would help. You soon realize that this learning/teaching is hard to limit to chess as it is already seen at the initial stage of learning.

Even the first (*ergo-trivial*) step – learning the *rules* how pieces move on the board will be virtually insurmountable in isolation, since these rules cannot be guessed on the basis of a non-exhaustive list of examples, say, a thousand samples, unless, besides ergo, you have a simple and adequate representation of the geometry of the chess board in your head.

If your are blind to the symmetries of the chessboard, the number of possible moves by a white rook ♖ in the presence of the white king ♔, that you must learn (in $64 \cdot 63$ positions), is $\approx 64 \cdot 63 \cdot 13 \approx 50\,000$. "Understanding" space with its symmetries, be this "understanding" preprogrammed or acquired by *a learning process of spatial structure*(*s*), is a necessary prerequisite not only for learning chess but also for communication/absorption of the rough idea of chess.[91]

But if you have no ergo counterparts to such concepts as "*some piece on a certain line*"[92] in your head, you'll need to be shown the admissible moves of the rook in *all* ($> 10^{45}$) possible chess positions.

And the more you think about it the clearer it becomes that the only realistic way to design a chess learning/understanding program goes via some general/universal mathematical theory equally applicable to learning chess and learning languages.

[91] The geometry of the board can be reconstructed from a moderate list of sample chess games with *Poincaré–Sturtevant space learning algorithms* but these algorithms are slow.

[92] Such "abstractions" are probably acquired by the visual ergo-system of a child well before anything as "concrete" as *white knight in a particular position on a chessboard*, for example.

13. Play, Humour, and Art

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Without play and "playful thinking" we would not be human.

Children carry magic lanterns within themselves – the world projects onto the playground screens in their minds. And a similar *play-mode behaviour* of kittens and puppies is familiar to all of us.

Most young mammals play and also some birds, e.g., crows and ravens. (It is not always clear what behavior can be classified as "play")

Playfulness is retained into adulthood in humans and dogs, and goes along with other neonatal characteristics. Some adults animals in the wild also play, e.g., *dholes*.[93]

A Bear and a Dog[94]

> [the dog] wagged his tail, grinned, and actually bowed to the bear, as if in invitation. The bear responded with enthusiastic body language and nonaggressive facial signals. These two normally antagonistic species were speaking the same language: "Let's play!" The romp was on. For several minutes dog and bear wrestled and cavorted.

There is no accepted adaptive evolutionary explanation for the play. Apparently, patterns of play programs reflect some facets in the mental architecture of the ergo-brain that came about *despite*, not because of, selection.[95]

Ego and Ergo in Play. The drive *to win* originates in ego, but "winning/losing" is, structurally speaking, a trivial component of play.

A pure ergo-system would not try to win but rather adjust to a weaker player to make the play/game *maximally interesting*.[96]

One's ego-mind approaches the problem of play (as much as everything else) with "why"-questions; purpose-oriented solutions are welcomed by ego and such "explanations" as *Oedipus complex* for chess are acceptable.

---

[93] Dholes, also called *red Asian dogs* and *whistling dogs*, are agile and intelligent animals, somewhat one-sidedly depicted by Kipling in *The Jungle Book*.

The systematic killing of dholes was conducted by locals and promoted by British sport hunters during the British Raj. Later, some European "naturalists" called for extermination of dholes, because dholes had no "redeeming feature" but rather hair between their toes. Despite recent measures protecting dholes their population (¡2000) keeps declining.

[94] Taken from: http://www.onbeing.org/program/play-spirit-and-character/feature/excerpt-animals-play/1070.

[95] Eventually, selection may win out and populate Earth exclusively with bacteria which would have no risk-prone inclination for play. This would be the most stable/probable state of the biosphere of an Earth-like planet, granted an "ensemble" of $10^{10^{10}}$ such planets.

[96] This may not be very *interesting* to the second player, e.g., in the cat and mouse game.

We – students of ergo – on the other hand, admit that we do not understand the deep nature of play, but we reject the very idea of any common sense (teleological) explanation.

For instance, contemplating on the "meaning" of a chess-like game, we do not care what drives one to win, but rather think of the architecture of an elaborate network of *interesting game positions*. Algorithms for representation of such networks lie at the core of *universal learning*.

> *The ... phrase that heralds new discoveries is not "Eureka!" but "That's funny".* Isaac Asimov[97]

Sense of humour, laughing at "funny", is closely associated with play – this is apparent in children. This "sense" is an instance of what we call an *ergo-mood* – a reaction of the ergo-brain to "*funny arrangements of ideas*".

Making a universal style program recognizing such "funny arrangements", say on internet pages, seem easier than recognizing *interesting arrangements of pieces* on the chessboard.

Performing as well as fine Arts – theatre, dance, music, along with painting, sculpture, poetry – grew out of child's play on the ego-soil of the Human Mind, where *aesthetic perception* – feeling of beauty of nature and of artistic beauty – is shared within our minds with the *sense of opposite-sex beauty*.

Music, poetry, the architecture of plants, animals and cathedrals, kaleidoscopic symmetry of peacock's tails – all with no "reproduction" tag on them,[98] – trigger in us a feeling similar to that caused by attraction to the opposite sex.

But this may only distract you from what we want to understand. For instance,

what is in arts which is of "ergo" rather than of "ego", what are universal structures in there?

Well, ... the formal study of arts, especially of music, goes back to Pythagoras.

Also, there are active fields of *neuroscience of art, neuroaesthetics, cognitive neuroscience of music*, with many publications openly accessible on the web.

For instance,

"*positron emission tomography scanning, combined with psychophysiological measures of autonomic nervous system activity*

shows that

endogenous dopamine release in the striatum at peak emotional arousal during music listening".[99]

And an avalanche of superlatives that music lovers pour on you when they speak of music[100] tells you something about the levels of endorphins released into their blood, but does not help answering the following kind of questions.

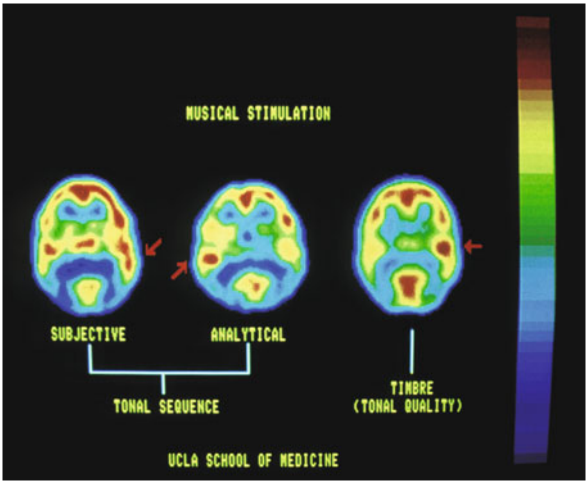
---

[97] Fifty-fifty maybe?

[98] Peacock's tails are sexually significant for peahens.

[99] See http://www.ncbi.nlm.nih.gov/pubmed/21217764.

[100] This is also how mathematicians speak of their beloved science.



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What is the starting level of complexity an ergo-system must have, such that, upon unsupervised learning, it will achieve the ability to "correctly" assign aesthetic values to pieces of art?

Probably, this level need not be prohibitively high, if such a "value" is represented *not by a number* $V = V(A)$ assigned to a piece of art A but by a (partial?) order relation

$$V(A_1) >_c V(A_2), c \in C,$$

that depends on c taken from a set C of groups of art critics c.[101]

14. Ergo in Science

... a scientist ... is a curious man looking through the keyhole of nature.

Jacques Yves Cousteau

It is the harmony of the diverse parts, their symmetry,
their happy balance; in a word it is all that introduces order,
all that gives unity, that permits us to see clearly and
to comprehend at once both the ensemble and the details.

Henri Poincaré

Seduction by an unadulterated beauty of the world overrides the pragmatic dictum of evolution and lures us to chess, to arts, and to the ultimate human game: The mad pursuit of harmonious structures in science and mathematics.

[101] Owners of art galleries routinely solve the problem of assigning consumer-dependent prices $P_c(A)$ to pieces A of modern art.

This is "ergo" in the human character that shapes the mental set-up of a scientist. "Ergo" makes the very existence of science possible.

Henri Poincaré articulates this as follows.

> *The scientist does not study nature because it is useful to do so. He studies it ... because it is beautiful.* [It is] *intimate beauty which comes from the harmonious order of its parts, and which a pure intelligence can grasp.*

But – one objects – Poincaré was a high priest of pure thought. Would experimentalists agree?

The experimental scientist who single-handedly contributed most to our electricity-hungry industrial civilisation was Michael Faraday.[102] He writes:

> *It is the great beauty of our science, chemistry ..., that ... opens the doors to further and more abundant knowledge, overflowing with beauty and utility.*

Yet, medical researchers – doctors and inventors of drugs – were not playing "scientific curiosity games" but were driven by the concern for the well-being of their fellow humans. Weren't they?

Let us listen to what Alexander Fleming, who discovered penicillin, and Howard Florey, who brought penicillin to therapeutic use, say.

> Fleming: *I play with microbes. There are, of course, many rules in this play ... but ... it is very pleasant to break the rules ....*

> Florey: *This was an interesting scientific exercise, and because it was of some use in medicine is very gratifying, but this was not the reason that we started working on it.*

We are in no position to brush aside what these people say about science. Penicillin has saved about 100 million human lives.[103] Without either Fleming or Florey half of us would not be alive today and younger ones would not be even born.

> *Science produces ignorance, and ignorance fuels science.*
>
> Stuart Firestein[104]

> *Science never solves a problem without creating ten more.*
>
> Bernard Shaw

Shaw meant this as a mockery, but science *is* an art of *not* understanding. We strive to understand but we are not satisfied with kindergarten explanations like: *Teeth are for chewing* and *wings are for flying.* We search for *new, unknown,*

---

[102] Without him on the scene, world history would be shifted by a few years backward and would be, of course, quite different from what we know as our world today.

[103] By comparison, the number of victims of 20th-century "fighters for people's happiness" is estimated at 180 million–220 million.

[104] Firestein is a biologist. In his lab, they study *the vertebrate olfactory receptor neuron as a model for investigating general principles and mechanisms of signal transduction.*

invisible to understand[105] what we see in front of our eyes.[106] We are happy to discover ten new problem where originally we could discern only a single one.

A four-year-old asks:

> *Why is the grass green? Why do we breathe? Why is the water wet and stones are hard? Why do we not see in the dark? Why do we not fall upward?*

A *plain and reasonable man* of Lev Tolstoy would smile at the naivety of these questions but a 21st-century scientist would readily admit that the he/she understands none of these either, but he/she may continue with further and better-focused questions:

> *How does chlorophyll-assisted photosynthesis work?*
> *When and exactly how did the great oxygenation event occur?*
> *How does an animal cell transfer chemical energy of oxidation into mechanical energy?*
> *What is a workable microscopic model of liquid water?*
> *What is the nature of divergences in the quantum electrodynamics model of light and matter?*
> *Is there a self consistent theory of quantum gravity? . . .*

Of course, *a plain and reasonable man* would have none of this. If anything, he would like to understand Nature in "a few simple words".

Well, the way the world is run may be beautifully simple, but our mind was not designed by Nature for contemplating her beauty. Perceiving this beauty requires an utmost intellectual effort on our part.

Even most familiar and apparently simple things in science are intuitively hard to accept, such as Newton's second law of motion that presents a manifestly mathematical (ergo)way of thinking about motion:

Lex II: Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.

This law, even more so than the first law, runs against how our visual and somatosensory (mainly *proprioception* – the body sense) systems represent properties of motion in our mind.[107] The majority of us, even if we can correctly recite the three laws of motion, do not believe in these laws. We intuitively reject them

[105] Here, "to understand" refers to what is called "understanding" by students of natural science. This is different from "to understand" of mathematicians and has little in common with "to understand" of humanities scholars.

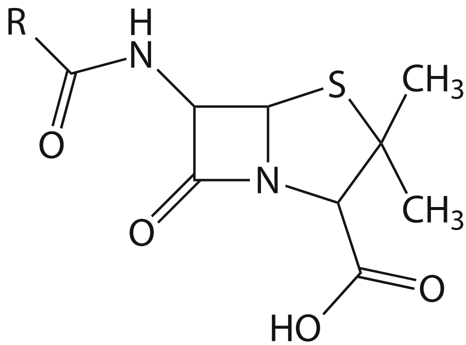
[106] A rare for humanities instance of an explanation of known by unknown, be it only a *conjectural* one, is that by Julian Jaynes who suggested that the major Mediterranean religions had resulted from *the Breakdown of the Bicameral Mind* about three thousand years ago. Jaynes' bicameral mind conjecture is, in principal, falsifiable – it may be either true or false. (One cannot assign truth value to "teeth are for chewing" theories of religion or of anything else for this matter.) But conducting an actual experiment verifying Jaynes' conjecture is ethically prohibitive.

[107] The essential logic of this reconstruction is of ergo but it serves the survival of our ego and serves it well, better than a mathematical Newtonian model would do.

in view of the apparent inconsistency of these laws with much of what we see with our own eyes, such as the motion of a pendulum that visibly contradicts the law of *conservation of momentum*.

But the main reason our brains resist absorbing scientific knowledge is the complexity of the combinatorial organisation of this "knowledge" that a brain needs to represent within itself.[108] Probably, the number of synaptic connections needed for *understanding* "N units of knowledge", of, say, *string theory* in mathematical physics grows (at least) as N^2 rather than const $\cdot N$ which is required for *absorbing*[109] the same number of "units of knowledge", say, in *cultural anthropology* that conveniently fit into pre-prepared niches in your ego-mind.

15. Unreasonable Men and Alternative Histories



Graphic formula of penicillin, where R is the variable group.

> *It* [science] *triumphantly tells him: how many million miles it is from the earth to the sun; at what rate light travels through space; how many million vibrations of ether per second are caused by light, and how many vibrations of air by sound; it tells of the chemical components of the Milky Way, of a new element – Helium – of micro-organisms and their excrements, of the points on the hand at which electricity collects, of X-rays, and similar things.*
>
> *But I don't want any of those things, says a plain and reasonable man – I want to know how to live.*
>
> Lev Tolstoy

[108]Partly, the difficulty in understanding "abstract" mathematical ideas is due to the protective wall separating ego from ergo.

[109]Memory in the brain is not straightforward unlike that on the magnetic tape. For instance, remembering long loosely structured (quasirandom) sequences, such as pages of telephone directories and arrays of the dates of "great historical events", is difficult – *excruciatingly difficult* for the mathematically inclined among us.

Nothing about penicillin – the miracle drug that had cured millions over millions of people was ever reasonable. And those who had discovered and developed this drug were anything but *plain and reasonable people.*

The modern chapter of penicillin's history starts with a *Staphylococci plate* in Fleming's lab that, between 27 July and 6 August of 1928, was unreasonably and unaccountably contaminated by an unusual (in its antibacterial activity) strain of *Penicillium notatum*, probably by the spores that escaped from the lab of La Touche – the mycologist at St Mary's Hospital working in the room below Fleming.

In 1928–1929, Fleming determined that the secretion from the mold *Penicillium notatum*, that he called *penicillin*, was effective against many bacteria.

(Penicillin suppresses growth of the so-called *gram-positive* bacteria, e.g., *streptococci* and *staphylococci*, that have no outer cell protective membrane, by blocking cell wall growth when bacteria replicate.)

Fleming suggested that penicillin could serve as a low toxicity[110] disinfectant; also he indicated the laboratory use of penicillin for the isolation of *Bacillus influenzae.* But despite a few therapeutic successes,[111] he became disappointed with instability of penicillin and turned to his other projects.[112]

In 1938, Ernst Chain (1906–1979), upon reading the 1929 paper by Alexander Fleming, proposed the study of penicillin to Florey.

In 1939, Howard Florey (1898–1968) created and directed a team of scientists for the study of anti-bacterial substances that are produced by mould. Using the sample of Penicillium notatum preserved by Fleming, they extracted and purified *penicillin* – the active antibacterial agent in the mould – and produced therapeutically significant amounts of it (1940), with the key roles played by three biochemists: Chain, Heatley,[113]and Abraham.[114]

---

[110]Penicillin kills bacteria but it is harmless for humans. However it is toxic to a few mammalians, e.g., to Guinea pigs, whose intestines are inhabited by gram-positive bacteria. Luckily, Fleming and people from the Florey's team, Chain and Heatley, tried penicillin on mice, apparently because they had limited amounts of the drug.

[111]In 1930, Cecil George Paine, treated a gonococcal infection in infants and achieved the first recorded cure with penicillin. He then cured four additional patients of eye infections, and failed to cure a fifth.

[112]Alexander Fleming (1881–1955) was a member of the research department at St Mary's Hospital in London that was organized and directed by Almroth Edward Wright (1861–1947).

In the first year of Wold War 1, Wright and Fleming worked out a treatment of infected wounds; ever since Fleming had been searching for antibacterial agents. In 1921, prior to penicillin, he discovered *Lysozyme* – an enzyme, present in tears, saliva, human milk, and mucus – that protects from gram-positive pathogens.

[113]Norman Heatly (1911–2004) devised the main steps for producing therapeutic quantities of penicillin. This, combined with know-how in fermentation technology of organic acids, has led to fast development of production of penicillin on an industrial scale.

[114]In 1943, Edward Abraham (1913–1999) determined the structure of penicillin which involved a *beta-lactam ring*; this was confirmed in 1945 by Dorothy Hodgkin by X-crystallography. In the 1950s, Abraham essentially contributed to isolation and development of *cephalosporin*, that kills penicillin-resistant bacteria.

Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?
EDWARD LORENZ

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There had been several decisive moments in history when ideas, insights, and decisions by such people as Fleming, Florey, Chain, Heatley, and Abraham were turning the path taken by humanity to its present course.[115]

Below are two other, even more vivid instances of (potential) instability of the course of the human history.

1. In 1896–1897, Ernest Duchesne (1874–1912) conducted the first(?) scientific study of antibacterial properties of mould. His results on *Penicillium glaucum*, similar to what was observed by Fleming, were recorded in his 1897 thesis: *Contribution à l'étude de la concurrence vitale chez les micro-organismes: antagonisme entre les moisissures et les microbes*, that he sent to *the Pasteur Institute.*

If Duchesne's work had been taken seriously by people (a single person?) at Pasteur Institute the development of antibiotics (and of the world pharmacological industry, in general) could have started a few decades earlier.[116]

It is hard to imagine what kind of world we would be then living in today.

[115] The perturbative effects of these people on human history were more subtle than of those depicted by Stefan Zweig in *Sternstunden der Menschheit*.

[116] This may be compared to what could have happened if Nägeli had understood Mendel. Possibly, nothing would have changed in both cases: The whole of the scientific community, stabilized by inertia, was no ready for these ideas.

2. A boy fails to recall the name of the river the city Berlin is located on and as a result he is denied entrance to a gymnasium in Odessa. Seeking education, he is compelled to move to the United States, where a few decades later he starts research on soil bacteria. Around 1940 this "boy" develops a comprehensive program for screening and testing *actinomycetes* for antibacterial activity. This leads to the discovery of a dozen antibiotics including, in 1943, *streptomycin* – a drug harmful to *gram-negative* bacteria, the first one effective against *mycobacteria* that cause tuberculosis.[117]

The name of the "boy" was Selman Waksman (1888–1973).[118] Streptomycin was discovered by Albert Schatz (1920–2005) who worked in Waksman's group.

Were Waksman's examiner in Odessa less pedantic, the discovery of streptomycin and other antibiotics could have been delayed by several years and hundreds of thousand people would have died of tuberculosis during this time.

A Few References On Antibiotics and their Discoverers

Gwyn Macfarlane, a haematologist who worked with Florey, wrote down two masterful accounts on the lives and works of the discovers of penicillin, [21, 22].

More can be found in [4, 6–8, 11, 12, 17, 24, 28, 30, 31, 36, 37].

16. Mathematics and its Limits

... the object of pure mathematics that of unfolding the laws of human intelligence.

James Joseph Sylvester

© William van der Fliet

[117] A hundred years ago, 10–15% deaths in Europe were inflicted by tuberculosis. In 2013, about nine million people in the world fell ill and 1.5 million died from TB.

[118] A significant factor turning Waksman's interest toward antibiotics, besides the achievements by Florey's group in Oxford, was the work of René Dubos (1901–1982) who, in 1939, under the influence of Oswald Avery (1877–1955) and with the help of Rollin Hotchkiss (1911–2004) at Rockefeller University in New York, isolated a bactericidal substance from the spore-forming bacillus of the soil, that he called *gramicidin*.

We share our inborn[119] ability to count with *pigeons* and *vervet monkeys*, but it is a long way from *quantity and shape perception* by animals (humans included) to something like *Ramanujan's mysterious formula*

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}} = \frac{2\sqrt{2}}{9801}\left(1103 + \frac{24\cdot 27493}{396^4} + \cdots\right)$$

This formula, similarly to the equally incredulous but more familiar *Leibniz formula* $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\frac{\pi}{4}$, relates the *geometrically* defined number $\pi = 3.14159265\ldots$ to an *arithmetically generated* infinite sum, that in the Ramanujan case is comprised of impossibly complicated terms.[120]

What allows such miracles in mathematics?
What is mathematics from the ergo perspective?
What is the mathematics that underlies ergo?

The relations between mathematics and ergo, that are by necessity circular, may be summarized as follows.

Mathematics at its core, is "just" an instance of an ergo-structure.
Mathematics is "just" a fragment of the collective[121] human ergo.
Mathematics is the tool and the language for a study of ergo-structures where the latter are "just" particular mathematical structures.

Let us say a few more words about these.

The shape of heaven is of necessity spherical. ARISTOTLE

1. *Core mathematics* is all about *amazing structures* clustered around *symmetries*: perfect symmetries, hidden symmetries, supersymmetries, partial symmetries, broken symmetries, generalized symmetries, linearized symmetries, stochastic symmetries. Two thirds of this core along with most theoretical physics would collapse if the symmetry axes were removed.

The main transitions in the evolution of mathematics were not achieved by reduction of unknown to known but, contrary to common sense, by inventions of "irreal objects" such as *negative* and later *complex imaginary* numbers, *infinitesimals*, *ideal numbers*, *n-dimensional spaces*, etc., similarly to how progress in physics was driven by "irreal ideas" of *atoms, wave functions, quantum fields*.

[119] Inborn? – not quite, at least not quite inborn in *humans*. Our number sense is intertwined with the language learned in the cradle.

[120] The series $1103 + \frac{4!(1103+26390)}{396^4} + \frac{8!(1103+26390\cdot 2)}{(2!)^4 396^8} + \frac{12!(1103+26390\cdot 3)}{(3!)^4 396^{12}} + \cdots$, unlike Leibniz' $1-\frac{1}{3}+\frac{1}{5}-\cdots$, converges exponentially fast; this allows a practical computation of the decimals of π with Ramanujan's formula.

[121] We mathematicians are a tiny community, something of order 0.001% of the total population, and, probably only a couple of hundreds among us (myself is not in the club) understand the Ramanujan formula.

It is amazing how mathematics manages to contain these symmetries and to represent, for example, "*roundness*" immanent in something like $\pi = 3.14159265\ldots$ by a Ramanujan-like, let it be infinite; yet, *arithmetic* formula or in the combinatorial, nearly digitalized, form of axioms, lemmas, theorems, and proofs.

This may seem not surprising, since the (collective) ergo-brain that created mathematics – represents the external world in this manner. But it may be *an endogenous* property of mathematics as well.[122]

This "combinatorial" nature of mathematics may also be compared to that used by Life for encoding shapes of organisms by DNA sequences, except that there is no (?) mathematical counterpart to *transfer of information by 3D-folding*. (A primitive form of "embryonal development" may be discerned in organisation of some mathematical proofs.)

A mathematician is an ergo-brain's way of talking to itself.

NIELS BOHR [misquoted]

2. Mathematics is the last born child of the ergo-brain, its development is guided by our ergos.

Mathematics *shines in the mind of God*, as Kepler says, but we are no gods and our minds are not pure ergo, our thinking is permeated by ego that makes it hard for us to tell "true and interesting" from "important" and that makes the (ergo) right choices difficult.

In the eyes of the ego-mind, much of mathematics appears *abstract and difficult*, but what you see in front of your eyes is *simple and concrete*.

But this simplicity is deceptive: What your eyes "see" is *not* simple – it is an outcome of an elaborate image building by your visual ergo-system that is, probably, more abstract and difficult than most of our mathematics.

Compelled by our "ergos", we search for another kind of "simplicity" that is *beautiful and interesting* – not at all trivial; *trivial* bores us to death. We are thrilled when ego-mind's "simple and apparent" is explained in terms of "abstract and difficult", that may not, a priori, even exist.

Our mathematical diamonds have been polished and their edges sharpened – century after century, by scratching away layers of ego from their facets, especially for the last fifty years. Some of what came out of it may appear as "abstract nonsense" but, as Alexander Grothendieck says,

The introduction of the cipher 0 *or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps.*

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[122] The proposition – *mathematics exists as an independently entity* – may be understood only metaphorically. No conceivable experiment or argument would make it more or less feasible. But ... you cannot do mathematics if you do not believe in it. And this is also the way that physicists take *reality of the physical word*.

3. In building a mathematical frame for "ergo" we need to recognize what of our mathematics is ready to serve as "parts" of ergo-systems, what should be rejected and what needs to be made anew. And remember that

you cannot apply mathematics as long as words still becloud reality.

as Hermann Weyl said.

Our choice of the components of a logical reconstruction of human ergo-brain, call it $\mathcal{HEB}$, follows the criteria used everywhere in mathematics:

NATURALITY, UNIVERSALITY, LOGICAL PURITY, CHILDISH SIMPLICITY.

Mathematical universality of $\mathcal{HEB}$, in particular of its learning strategies (programs) may be seen in how we *enjoy and learn many different* logically complicated games. Thus, for instance a chess learning program in somebody's $\mathcal{HEB}$ must be a *specialization of a universal* learning program.

But why should such programs be simple? After all

The human brain is the most complicated object in the Universe. Isn't it?

The answer is that most general/universal theories are logically the simplest ones.[123] What is not simple is discovering and formulating such theories.

As mathematicians we are ready to accept that we are a hundred times stupider than evolution is but we do not take it for the reason that evolution is able to make miracles, such as a logically complicated brain at birth.

Believers in simplicity, we seek our own solution to the *universal learning problem* by adapting the *purest* kind of mathematics to the "dirty world" of *flows of signals* and their "*transformations*" by our (even if conjectural) ergo-brains.

But beware, mathematics that directs your thoughts toward "logically perfect structures", may mislead you when it comes to "real life".

(The arithmetic of numbers is seductively beautiful, but the surface temperatures of the sixteen named stars in the constellation of the Great Bear, even if taken in Kelvin, are not meant for addition and multiplication.)

The structures of "ergo" grow on the soil of mathematics but they diverge unrecognisably far from their pure mathematical prototypes.[124]

And it may appear – this has been the dominant philosophy in mainstream AI – that it is not so much mathematics proper, but rather such concepts as *axiomatic systems, automata, Turing machines, Gödel's theorems, etc.*, that are essential for understanding mental processes.

However, these ideas – this is witnessed by their poor record in implementing Turing's original program – are as insufficient for understanding the nature of human thinking processes as for illuminating the nature of human mathematics.

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[123]The simplicity of a universal idea, e.g., of *Gödel's incompleteness theorem*, may be obscured by a plethora of technical details.

[124]This is not so in theoretical physics but similar to the position of mathematics in shaping (mostly unknown) fundamental principles of biology.

17. Numbers, Symmetries, and Categories

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The existence of Mathematics as we know it strikes one as improbable as the emergence of Life on Earth. Nothing in the foundation of mathematics suggests such a thing is possible, like nothing in Earth's chemistry suggests it can beget Life.

One may say that mathematics starts with *numbers*. We are so used to the idea that we forget how *incredible* properties of *real numbers* are. The seamless agreement of several different structures – *continuity, order, addition, multiplication, division* – embodied in this *single* concept is amazing.

Unbelievably perfect *symmetries* in geometry and physics – *Lie groups, Hilbert spaces, gauge theories...* – emerge in the world of numbers from the seed of the Pythagorean theorem. Mathematics and theoretical physics are the two facets of these symmetries that are both expressed in essentially the same mathematical language.

As Poincaré says,

> *... without this language most of the intimate analogies of things would forever have remained unknown to us; and we would never have had knowledge of the internal harmony of the world, which is, as we shall see, the only true objective reality.*

In the "harsh real world", away from pure mathematics and theoretical physics, the harmony of the full "symmetry spectrum" of numbers comes into

play only rarely. It may even seem that there are several different kinds of numbers: some may be good for *ordering* objects according to their size and some may be used for *addition* of measured quantities. Using the all-powerful real numbers for a limited purposed may strike you as wasteful and unnatural.

For example, *positive* numbers appear in classical physics as *masses* of bulks of matter but electric charges represent positive and negative numbers. The relevant *operation* with these numbers is *addition*, since mass and electric charge are naturally (nearly perfectly) additive: $(a, b) \mapsto a + b$ corresponds to bringing two physical objects together and making a single $(a + b)$-object out of the two corresponding to a and to b.

But there is no comparably simple implementation of, say, $a \mapsto 2a$ – one cannot just copy or double a physical object.[125] And writing $2a = a + b$ for $a = b$ does not help, since mutually equal macroscopic physical objects do not come by themselves in physics.

In contrast, doubling is seen everywhere in Life. All of us, most likely, descend from a polynucleotide molecule that had successfully doubled about four billion years ago. Organisms grow and propagate by doubling of cells. Evolution is driven by doublings of genomes and of significant segments of whole genomes (not by so-called “small random variations”).

A true numerical addition may be rarely (ever?) seen in *biology proper* but, for example, additivity of electric charges in neurons is essential in the function of the brain. This underlies most mathematical models of the neurobrain, even the crudest ones such as neural networks. But the ergo-brain has little to do with additivity and linearity.[126]

The apparent simplicity of *real numbers* represented by points on an infinite straight line is as illusory as that of visual images of the “real world” in front of us. An accepted detailed exposition (due to Edmund Landau) of real numbers by *Dedekind cuts* (that relies on the order structure) takes about a hundred pages. In his book *On Numbers and Games*, John Conway observes (and we trust him) that such an exposition needs another couple hundred pages to become complete.

To appreciate this “problem with numbers”, try to “explain” real numbers to a computer, without ever saying “obviously” and not resorting to anything as artificial as decimal/binary expansions. Such an “explanation computer program” will go for pages and pages with a little bug on every second page.

We shall not attempt to incorporate the full theory of real numbers in all its glory into our ergo-systems, but some “facets of numbers” will be of use. For example we shall endow an ergo-learner with the ability of distinguishing frequent and rare events, such as seen in the behaviour of a baby animal that learns not to fear *frequently* observed shapes.

---

[125] Certain power/energy quantities, e.g., those measured in *decibels*, are described on a multiplicative (logarithmic) scale, and their doubling is perfectly meaningful.

[126] “*Non-linear*” customarily applies to systems that are set into the framework of numbers with their *addition structure* being arbitrarily and unnaturally contorted.

On the other hand, while describing and analyzing such systems we shall use real numbers as much as we want.

Numbers are not in your ergo-brain but the idea of symmetry is in there. Much of it concerns the symmetries of our (Euclidean) 3-space, the essential ingredient of which – the group of the (*three-dimensional Lie*) *group* of all possible motions, call them *rotations* of the Euclidean *round* 2-*sphere* within itself – has been fascinating mathematicians and philosophers for millennia. And not only *the haven* of Aristotle but also your eyes and some of your skeletal joints that "talk" to the brain are spherical; hence, *rotationally symmetric*.

Building and identifying symmetries *within itself* serves as an essential guideline for an ergo-learner. These are created, seen from outside, by a *statistical analysis*[127] *of signals* that *break* spatial and temporal symmetries.

For instance, the input of the visual system may be represented by the set of samples of a distribution of (not quite) a probability measure on the set of subsets of the light receptors in your retina. *In principle*, this is sufficient for reconstruction of Euclidean geometry similarly to how Alfred Sturtevant obtained the partial genetic map of a drosophila X chromosome on the basis of distributions of phenotype linkages.

However, your brain would not (?) be able to "map" the 3D-space without receiving along with visual signals also signals coming from the firing of the motor neurons controlling your motions, especially those of your eyes, which, being spherical, enjoy full rotational symmetry.[128] The movements of the eye, have, apparently, prepared the brain for the idea of spatial symmetries and have helped the brain to learn how to identify images that move in the visual field.[129]

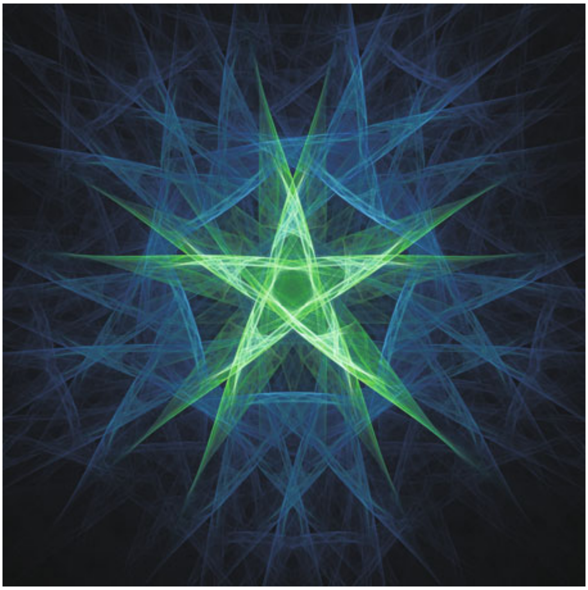
Our ergo-brain is also sensitive to *arithmetic symmetries* that issue from prime numbers as is seen in the recurrence of the *magical pentagram figure* depicting what mathematicians called *five-element Galois field* that can be visualized as the set of the five vertices of ⬠ with 20 transformations acting on it, where only 10 of them are geometrically apparent – [5 rotations] × [2 reflections]. But there is an extra one, that is there because 5 is *a prime* number and that can be depicted as ⬠ $\mapsto$ ⛤.

A fantastic vision, unimaginable to ancient mystics and to mediaeval occultists, emerges in the *Langlands correspondence* between arithmetic symmetries and the *Galois symmetries* of algebraic equations, where much of it is still in the

[127]The initial learning programs we envisage contain no counting mechanism, but an ergo-learner must be able to distinguish "significant/persistent" signals and ignore "accidental" ones.

[128]The eye shares spherical symmetry with the shoulder and hip joints that also have three degrees of freedom, while the cylindrical knee joint allows only circular motions. And the elbow hinge-joints is "designed" with *exactly two* degrees of freedom.

[129]This is explained in ○IV of Poincaré, *La science et l'hypothèse*.



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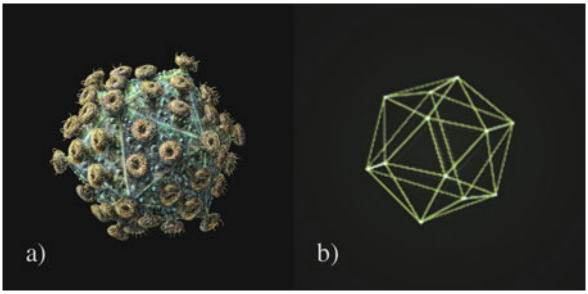
clouds of conjectures. It is tantalizing to trace the route by which the ergo-brain has arrived at comprehension of these kinds of symmetries.[130]



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Categories, Functors, and Meaning. Mental ergo-objects, e.g., sentences of a language, rarely (ever?) possess perfect internal symmetries, that move objets within

[130]Seen by an outsider, the symmetry in mathematics is diluted to the point of invisibility by *useful formulas, difficult computations, efficient algorithms, logical axioms, reliable* (*or unreliable*) *statistics* . . . .

themselves, such as the rotations of spheres, pentagons, or of icosahedra[131] preserving their geometric structures. (An icosahedron admits 120 transformations where – this is not fully accidental – $120 = 5! = 1 \times 2 \times 3 \times 4 \times 5$.)

However, certain transformations of such objects, e.g., of sentences, can be depicted, albeit only approximately, in *mathematical category theory.*

The simplest, geometric rather than syntactic, transformations are *insertions* of strings of letters into longer strings positioned somewhere else in a text,

$$\dots ABC \dots \longrightarrow \dots DEABCMN \dots .$$

Mathematical categories extends the concept of symmetries by allowing the kind of transformations that "move" one object into another one.

Such transformations are depicted by arrows between objects, e.g.,

$$\text{string}_1 \to \text{string}_2, \text{ or symbolically } \circ \xrightarrow{r} \ast,$$

where the structure of a particular category is given by a *composition rule* between incoming and outgoing arrows for all objects in this category.

Namely, one distinguished certain triples of arrows, say (r, s, t) between triples of objects, call them $(\circ, \ast, \star)$, such that $\circ \xrightarrow{r} \ast$, $\ast \xrightarrow{s} \star$ $\circ \xrightarrow{t} \star$, for which one declares that t equals the composition of r and s and one depicts this by the following diagram.

$$\begin{tikzcd} \circ \arrow[r, "r"] \arrow[dr, "t"'] & \ast \arrow[d, "s"] \\ & \star \end{tikzcd}$$

(In geographic terms, objects are "locations" and arrows between pairs of "locations" – there may be many of these – are possible routes from one location to another one, where composition of routes is the route that is obtained by consecutively following these routes, say, first r from $\circ$ to $\ast$ and then s from $\ast$ to $\star$.)

The combinatorics of (large) arrangements made from triangles of arrows carries an unexpectedly rich amount of information about the *internal* structures of the objects in our category. For instance, a seemingly vague idea of "*naturality*" of a certain mathematical construction can be non-ambiguously expressed in terms of *functors* between categories.

When applied to a language, this leads to a workable (sometimes called "holistic", see p. 242 in [20]) definition of "meaning" of a text without any reference to "real meaning" along the ideas maintained by Zelig Harris: *The meaning of a word is determined by statistics of distributions of accompanying words.*

---

[131] Viruses – they are not bound by ergo – love icosahedral symmetry, since this minimises the area of their protein shells that must contain all of the DNA encoding these very proteins.

This is how Life works: Information and geometry (of physical matter) go hand in hand. But mathematicians lag far behind viruses in the solution of *inforimetric problems.*

18. Logic and the Illusion of Rigor

As we aim at the very source of mathematics – ergo-brain itself – and try to develop a theory of ergo-systems, the purity and simplicity of the building blocks of such theory becomes essential. It is not logical rigor and technical details that are at stake – without clarity you miss diamonds – they do not shine in the fog of an ego-pervaded environment.

The evolution of mathematical concepts in their convergence to the clear shapes they acquired in the 21st century suggests how one may design ergo-systems. Yet, not all roads we explored had lead us to the promised land; understanding what did not work and why may be more instructive than celebrating our successes.

The problem with our mathematical ideas is not that they are too abstract, too difficult. or too farfetched, but that we lack imagination for pulling abstract difficult and farfetched ideas out of thin air. Nor do we have the foresight for predicting how an idea will develop.

> *Contrariwise, if it was so, it might be;*
> *and if it were so, it would be;*
> *but as it isn't, it ain't.*
> *That's logic.* LEWIS CARROLL

According to the *logicism* of Frege, Dedekind, Russell, and Whitehead, mathematics is composed of atomic *laws of thought* dictated by formal logic and the rigor of formal logic is indispensable for making valid mathematical constructions and correct definitions.

Admittedly, logicians participated in dusting dark corners in the foundations of mathematics but ... most mathematicians have no ear for tunes of formal logic.[132] We are suspicious of "intuitive mathematical truth" and we do not trust *meta*mathematical rigor[133] of formal logic.

(Logicians themselves are distrustful one of another. For example, Bertrand Russell pointed out that Frege's *Basic Law V* was self-contradictory, and in Gödel's words,

> [Russell's] *presentation ... so greatly lacking in formal precision in the foundations ... presents in this respect a considerable step backwards as compared with Frege.*

---

[132]Not everything in logic is collecting, cleaning, and classifying morsels of common sense – it is hard to believe but logical thinking can be creative. But this "creative logic" is what we call *mathematics*. We happily embrace *model theory*, *set theory*, *theory of algorithms*, and other logical theories that became parts of mathematics.

[133]The concept of *logical rigor* unlike that of *mathematical rigor* cannot be defined even with minimal requirements for being precise and rigorous.

Russell's words

> *Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.*

apply to formal logic rather than to mathematics.)

The cleanness of things does not make them beautiful in the eyes of a mathematician. We care for logical pedantry as much as a poet does for preaching of grammarians.

The soundness of mathematics is certified by an ***unbelievably equilibrated*** *harmony* of its edifices rather than by the strictness of the construction safety rules. The miracle of the Leibniz formula $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$ (of 1682) achieved by an appeal to infinitesimals, makes the criticism of insufficient rigor in mathematics by George Berkeley (1734) as well as the idea of "redemption" of Leibniz' calculus by Abraham Robinson (1966) look puny.[134]

Historically, the system of calculus rolling on fuzzy wheels of *infinity* and *infinitesimals*, has been the main intellectual force driving the development of mathematics and science for more than three hundred years. But just a step away from mathematics, volumes of philosophical speculations on the "true nature" of infinity remain on library shelves covered in dust year after year.

(Unbelievably, as recently as at the beginning of the 20th century, Florian Cajori, then a leading historian of mathematics, hailed *The Analyst* – the treatise by George Berkeley who had lambasted "non-rigorous infinitesimals" – as *the most spectacular event of the* [18th] *century in the history of British mathematics.*

The landscape of 18th-century British mathematics was, indeed, so bleak that even *The Analyst* was noticeable. But there were, however, two English mathematicians who, unlike Berkeley, had left non-trivial imprints on 18th-century science – Thomas Bayes, who suggested what is now called a *Bayesian* approach to empirical probability,[135] and Edmond Halley, famous for computing the orbit of *Halley's Comet.*[136])

We cannot take seriously anything like $(a, b) := \{\{a\}, \{a, b\}\}$.[137] To get "convinced" that this definition is worth making, you must accept logicians' appeal to metamathematical intuition; however, *nothing* of meaning can be communicated *without a use of natural language*, the metaphoric essence of which dissolves logicians' idea of "perfect rigor".

---

[134] The achievement of Robinson from a working mathematician's perspective was not so much in justification of Leibniz' idea of infinitesimals but rather in a vast and powerful extension of this idea.

[135] The Bayesian approach relies on continuous updating of conditional probabilities of events rather then on integrated frequencies; it is systematically used nowadays in *machine learning*.

[136] Halley's is the only short-period comet that is clearly visible from Earth when it returns to the inner solar system, at approximately 75 year intervals.

[137] This is the 1921 definition of an *ordered pair* by Kuratowski.

We communicate mathematics not by copying scores of logical symbols from one mind to another but by making another's ergo-brain resonate to the tunes we hear within ourselves.

And we do not bow down before "power of intuition" so extolled by logicians and mathematicians but rather try to understand the source of this intuition in the human ergo-brain/mind.

Who argues establishing "truths" of certain kind needs perfect precision, all 200% of it; below is an example of such a "truth".

$$4579 + 8763 - 3459 + 4686 - 6537 + 7763 - 4579 + 1099 - 8765 + 1238 - 3677 = 1111.$$

But – this is a miracle from a logician's point of view – a formally imprecise outline of the "core idea" of a *meaningful* statement, even of a relatively structurally plain one such as *Gödel's first incompleteness theorem*, ascertains its validity in the eyes of a mathematician.

(It is how "*something alive*!" in an incomplete and distorted image of an animal, that may be even unknown to you, catches your eye.)

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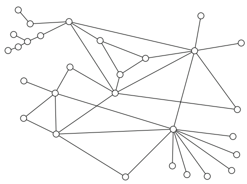
(The century-old foundational dust finds its way to our textbooks under the pretext of rigor,[138] e.g., in the following definition of a *graph* G as

an ordered [by whom?] *pair* $G = (V, E)$ *comprising a set* V *of vertices ...*

Brr ..., there is more sense in simply drawing a graph than giving such a definition.)



[138]The persistent urge for "rigor" in certain minds begs for an explanation by a Freudian style psychologist.

Problem with Order

The concept of *logical rigor* doesn't fare well in the ergo environment. For instance, the abstract concept of *order* cannot be defined logically rigorously but only with reference to a preexistent order in the physical or psychological medium where the idea of order is expressed.

Ordering a pair involves breaking $-1 \leftrightarrow +1$ symmetry similar to that encountered by *Buridan's Ass.* (To see it picturesquely, switch from the habitual $A \prec B$ to ▲ ❖ ▼ that is graphically but not contextually invariant under rotation by 180°.[139]

Besides the mathematical impossibility of algorithmically resolving the "order problem" (the reader is invited to count the number of logical flaws in the above "solution" $(a, b) := \{\{a\}, \{a, b\}\}$) there is good evidence that there is no innate "idea of order" in our ergo-brains. For instance, children often use mirror images of letter-signs when they begin writing and mathematicians tend to reverse (mentally as well as graphically) the directionality of their inequalities.

Accordingly, we should not (?) postulate *a primary idea* of order in our design of ergo-systems.

In general, we need to be choosy in our terminology and in assigning basic concepts/operations to our learning systems:

> *no elementary structure, no matter how simple looking and "obvious", can be taken for granted.*

For instance, a child who has learned to count on fingers and can figure out that $2 + 3 = 5$ and $2 \times 3 = 6$ resists $3 = 3$ and $1 \times 5 = 5$. If you think the child should be instructed to accept $[3 = 3] = [3 + 2 = 5]$, this is yourself, not the child, who lacks proper education: $[3 = 3] \neq [3 + 2 = 5]$ – mathematics has arrived at the point of accepting child's attitude and developing means (around *category theories*) for differentiating between various kinds of "equalities". And a *hierarchy of equalities* is essential for our models of ergo-brain.

Logic in Science

Mathematical rigor and logical certainty are absent not only from the logical foundations of mathematics, but also from all natural sciences even from theoretical physics. Einstein puts it in words:

> *As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*

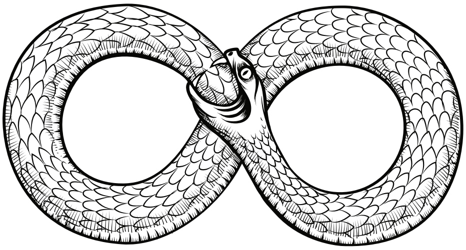
But "the physical level of rigor" is higher on certainty than the logical one, since reproducible experiments are more reliable than anybody's, be it Einstein's or Gödel's, intuition.

[139]One may only wonder what face of mathematics one would see in the world where "formulas" were invariably represented by *symmetric* combinatorial arrangements of symbols on the plane.

19. Infinite Inside, Finite Outside

If any philosopher had been asked for a definition of infinity,
he might have produced some unintelligible rigmarole,
but he would certainly not have been able to give a definition
that had any meaning at all.

BERTRAND RUSSELL



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Mathematics is the abode of infinity. Almost all our theorems are about infinite objects, e.g., associated with *real numbers*, such as *Lie groups*[140], or about infinities of finite objects, e.g., *prime numbers.*[141]

And basic properties of finite objects such as prime numbers can be understood only in the ambience of transcendental infinities of real as well as *p-adic numbers*[142] and of uncountably infinite *adelic groups*[143] represented by rotations of *infinite-dimensional geometric* (*Hilbert*) *spaces.*

Also this kind of infinity is indispensable for how we describe the physical universe and its fragments. Most (all?) physical systems are modelled by (uncountably) infinite sets of *real numbers.* (Physical constants, even dimensionless ones such as 0.00729735257[144] . . . , are, of course, not quite numbers, but it is hard to pinpoint what exactly this quite is.)

But looking from outside, the whole body of mathematics, call it $\mathcal{M}$, is a humble mathematical object describable in *finitely many* words.[145] These words

[140] The basic example of such a group is that of *rotations of an imaginable rigid body* in 3D-space.

[141] Even individual (especially transcendental) real numbers, such as
$\pi = 3.14159265358979323846264338327950288419716939937510582097494459230781640628 62\ldots$
would barely exist without an ocean of infinity that surrounds and supports them.

[142] These are limits of rational numbers where the usual rule $\varepsilon^i \to 0$ for $|\varepsilon| < 1$ and $i \to \infty$ is formally replaced by $p^i \to 0$ for a given prime number p and $i \to \infty$. Amazingly, this leads to a meaningful concept of a “number”.

[143] These groups are infinite products of certain Lie groups with all their p-adic counterparts, $p = 2, 3, 5, 7, 11, \ldots.$

[144] This is the *fine-structure constant* $\alpha = \pi\cdot[\textit{elementary charge}]^2/hc$, introduced by Sommerfeld in 1916 to account for the spectrum of the hydrogen atom.

[145] From a non-mathematical scientist’s point of view, such an $\mathcal{M}$ is *a mathematical model* (not of a kind defined by mathematical logicians) of “real mathematics” that is practiced by human mathematicians on the planet Earth.

however, generate a language that is represented by something *infinite*, call it $\mathcal{M}'$ – a "fragment" of $\mathcal{M}$ taken from it and positioned outside of it.

Thus, $\mathcal{M}$ starts looking like a giant kaleidoscope full of tiny mirrors $\mathcal{M}'$ each with the ability to fully reflect all of $\mathcal{M}$ with the mirrors, including $\mathcal{M}'$ itself, in it.

© Vadim Kaimanovich

The coexistence of "equivalence" $\mathcal{M}' \sim \mathcal{M}$ with *strict* "inclusion" $\mathcal{M}' \subsetneqq \mathcal{M}$ allowed Gödel to untangle self-referentiality within the *Liar Paradox: I am unprovable is unprovable* and, thus, proves his theorem on the existence of *formally unprovable mathematical propositions.*

(Gödel's $\mathcal{M}'$ is an infinite set of strings in finitely many, say 10, symbols/letters: These strings represent proofs in $\mathcal{M}$ described by sentences written in a language with a ten-letter alphabet. This $\mathcal{M}'$ "embeds" into $\mathcal{M}$ by means of *Gödel's enumeration*, where the "letters" are depicted as 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; thus, "translating" formulas in $\mathcal{M}'$ to *decimally represented* numbers from $\mathcal{M}$ and properties of these formulas, notably their provability/non-provability, to arithmetic properties of these numbers.[146]

Gödel's proof depends on the "rigid" physical space-time of our Universe that serves as supporting background for *positional* representation of numbers. No such proof would be possible in a "Liquid World" with *nothing* "rigid" like our space-time in it.)

All of mathematics with all these *uncountable sets* that are incomparably greater than $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ can be reflected in the countably large (small?) mirror of arithmetic according to a childishly simple theorem called *Skolem's paradox*:

> *"All" mathematical objects comprising uncountable sets can be "adequately represented" by countably many their "verbal descriptions".*

But this is still about "infinities"; in "reality", all these infinities are fictional: The number of mathematical statements that can ever be produced by intelligent

[146]Logicians, apparently to show their respect for Gödel, compose this with the map $\mathbb{N} \to \mathbb{N}$ defined by $\{a_i\} \to \prod_i p_i^{a_i}$ for the prime numbers $p_1 = 2, p_2 = 3, \dots, p_i, \dots$.

beings in our or any other conceivable universe is pathetically small[147] compared to the immeasurable infinity of "objects" allowed by the grammar of the mathematical language.

Also, since all physical, biological, and/or mental structures may function only on specific space/time/complexity scales, meaningful philosophical interpretations of Gödel's theorem seem difficult, if not impossible, to attain.

Most (all?) speculative (rather than purely mathematical) arguments about these structures (e.g., in the philosophy of artificial intelligence) with a whiff of a hint at Gödel's or similar "infinity theorems" such as *Turing halting theorem* or *Kolmogorov–Chaitin complexity* inevitably harbor gross misinterpretations of the concepts underlying these theorems.

Ergo-Logical Model of Mathematics

Out of an infinity of designs a mathematician chooses
one pattern for beauty's sake and pulls it down to earth.

Marston Morse

A logician's model $\mathcal{M}$ of mathematic, e.g., *Peano arithmetic*, has an unattractively amorphous structure that may be compared to that of the morass of *all positions* of chess pieces on the board that may be obtained from the initial position by the rules of chess.

(Peano arithmetic is an axiomatic description of the structure of integers $n = 1, 2, 3, \dots$ that is defined via the *successor operation* $n \mapsto n'$ (where, secretly, $n' = n + 1$),

$$\overset{1}{\bullet}\mapsto\bullet\mapsto\cdots,$$

where the basic axiom is that of *induction*:

If a "proposition (or a definition) expressed by a formula" F is valid for $n = 1$ and F-for-n implies F-for-n', then F is true for all n.

For instance the addition $m + n$ is *defined* by the formula

$$m + 1 = m' \;\&\; m + n' = (m + n)'.$$

There is no means in the traditional logic to see what is "interesting" and to describe the structure of "the set" $\mathcal{M}^{\mathrm{ergo}}$ of all "interesting" theorems.[148]

One of the goals of ergo-logic is to achieve this end but we cannot say yet with a convincing degree of precision what *interesting* theorems (and/interesting chess positions) are, but there are illuminating examples.[149]

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[147] The number of strings of symbols that can be generated during the life cycle of any conceivable universe can be safely bounded by $10^{10^{10}}$.

[148] "Interesting" theorems do not naturally comprise what we call "a set".

[149] It may also be worthwhile to borrow ideas from evolutionary biology (and immunology?) and pursue the ideas of *competition, selection, adaptation* (partly) responsible for the emergence of "interesting structures" in living systems and, on a smaller scale, in combinatorial games such as chess.

For instance, Harrington and Paris found out several bona fide mathematical propositions, such as *Goodstein's base change theorem* that are unprovable in Peano arithmetic.[150]

20. Small, Large, Inaccessible

Mathematicians treat all numbers on equal footing, be these

$$2,\ 3,\ 4,$$

or

$$10,\ 20,\ 30,\ 40,$$

or

$$1\,000,\ 10\,000,\ 100\,000,\ 1\,000\,000,$$

or

$$10^{10},\ 10^{20},\ 10^{30},\ 10^{40},$$

or

$$10^{10^2},\ 10^{10^{30}},\ 10^{10^{400}}$$

or

$$2^{3^{4^{\cdot^{\cdot^{\cdot^{399^{400}}}}}}}$$

But "democracy of numbers" breaks down in the "real world", be it the physical Universe or the human ergo-world.

Partly this is because properties of numbers (and formulas in general) depend on how their representations by symbols spreads over the background either in physical space/time or in the human ergo-mind.

Even quite small numbers, in fact, everything above four, unless they are represented by "structured entities" in some way, are not accessible to the human (ergo)mind.

Non-accidentally, the grammars of some languages, e.g., of Russian, distinguish the numbers 2, 3, and 4, while 5, 6, ... , 20, 30, 40, ... , 100, 200, ... (but not, say 23, 101, and 202) are, syntactically speaking, go in the same basket as infinity.[151]

Yet, some psychologists maintain the idea that a few larger numbers

$$\bullet\bullet\bullet\bullet\bullet, \quad \bullet\bullet\bullet\bullet\bullet\bullet \quad \text{and} \quad \bullet\bullet\bullet\bullet\bullet\bullet\bullet$$

[150] See *Brief introduction to unprovability*, by A Bovykin at https://www.cs.umd.edu/~gasarch/TOPICS/largeramsey/bovINTRO.pdf and the next section.

[151] Amusingly, there is also a chasm in essential properties between geometric spaces of dimensions 1, 2, 3, 4, and those of dimension 5 and more.

are also humanly perceptible but nobody(?) would claim he/she can immediately grasp

●● or even ●●●●●●●●●●.

Habituation to the decimal notation deceives us. We have no idea what such numbers as 65 536, even less so 7 625 597 484 987, are[152] unless we write them as $2^{2^{2^2}}$ and 3^{3^3}.

Even recognition of "*seven trillion*" in 7 625 597 484 987, unlike that of "*million*" in 1 048 756$\left(= 2^{2^{2^2}+2^2}\right)$, needs a few rounds of the mental counting algorithm that is not innate to the human mind.

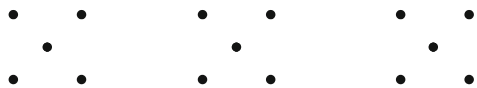
On the other hand, the human (ergo)mind that feels uncomfortable with, say 177 148, readily accepts the same (?) number structurally organized as $3^{3^3+2} + 1$.

Probably, this is because the "*combinatorial geometry of the* (*ergo*)*mind*" is nearer to that of the arrangements of exp-towers in the 2-plane than to the uneventful linearity of ●●●●●●●●●●●●●●●●●●●●●... as well as of the ordinary positional depiction of numbers and formulas.

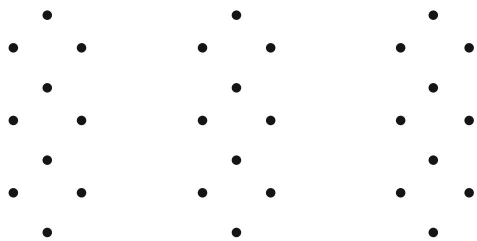
Look more closely at how it works. One instantaneously evaluates the cardinality of ●●●●, one needs a fraction[153] of a second to identify "(almost) unstructured five" ●●●●●, it takes a couple of seconds for ●●●●●●● (it is much faster if the symmetry is broken, for instance, as in ●●● ●●●●) and it is impossible with

●●●●●●●●●●●●●●●.

But a little structure helps:



And slightly larger numbers, such as



if perceived, then only through a lens of mathematics.

[152]Browsing through

$$\underbrace{\bullet\cdots\bullet}_{7625597484987}$$

point by point would occupy you for a couple of *hundred thousand* years.

[153]This may be a significant fraction, well above 200 milliseconds.

Our intuition does not work anymore when it comes to thousands, millions, billions. Answer fast:

Do you have more hairs on your head (assuming you are not bald) than the number of people an Olympic stadium may contain?[154]

What is greater, the number of bacteria living in your guts or the number of atoms in a bacterium?[155]

Below are ergo-relevant numbers.

• TIME. A hundred years contain < 3.2 billion seconds. With a rate three words per second you vocalize *less than ten billion* (10^{10}) words in the course of your life.

Ten billion garrulous individuals all together[156] will utter *at most*

$$10^{10} \times (3 \times 3.2 \cdot 10^7) \times 5 \cdot 10^9 < 5 \cdot 10^{27}$$

words until Sun turns into a *red giant* in about five billion years.

Speaking more realistically, humanity *cannot* come up with more than 10^{12}–10^{18} *different ideas* – poems, theorems, computer programs, descriptions of particular numbers, etc.[157]

10^{15} years of possible duration of the Universe is made of less than $10^{46} = 10^{15} \times 3 \cdot 10^7 \times \frac{1}{3}10^{24}$ *jiffy-moments.*[158]

• BRAIN. The number of neurons in the human brain is estimated between ten and a hundred billion neurons with hundreds of synaptic connections per neuron, somewhere between 10^{12}–10^{14} synapses all together.

This gives an idea of the volume of the memory stored in the brain, that is comparable to that on a computer hard disk of about 10^{12}–10^{13} bits.

The (short time) brain performance is limited by the *firing rates* of neurons – something about 100 times per second.[159] Thus, say hundred million active neurons can perform 10^{10} "elementary operations" per second[160], which is what an average computer does.[161]

• LANGUAGE. There are 10^{22}–10^{25} grammatical sentences in five words in English.

---

[154] Both numbers are about 100 000.

[155] There about 10^{11}–10^{14} atoms in bacteria and more than 10^{12}–10^{13} bacteria living in your body, mainly in your guts.

[156] The human population on Earth today is slightly above seven billion.

[157] LIFE on Earth, in the course of its $\approx 3.9 \cdot 10^9$ year history, has generated a comparable number of "ideas" and recorded them in DNA sequences of organisms inhabiting the planet.

[158] *Jiffy* $\approx 3 \cdot 10^{-24} s$ is the time needed for light to travel a proton-sized distance.

[159] It takes 8 milliseconds for the brain of a *star-nosed mole* to decide what is edible and what is not. This is fifty times faster than it takes for a driver to decide whether to brake or accelerate on yellow.

[160] But the rate of learning is measured not in seconds but in hours, days, months, years. This is so, partly, because modification of the strength of synaptic connections is slow.

[161] The speed of modern *supercomputers* is measured in *petaflops* corresponding to 10^{15} (*floating point*) operations per second. This is achieved with particularly designed network architectures of processors that allow thousands (not millions as in the brain) of operations performed *in parallel.*

• Space. A *glass of water* contains about 10^{25} molecules, the *planet Earth* is composed of about 10^{50} atoms, and the *astronomically observable universe* contains, one estimates today, 10^{80} particles.[162]

Thus, there are (significantly) less than 10^{130} *classical* (as opposed to quantum) "events" within our space-time and this grossly overestimated number makes

> *an unquestionable bound of what will ever be achieved by any conceivable (non quantum) computational/thinking device of the size of the Universe.*

But ... there are at least $2^{10^{10}} > 10^{3\,000\,00000} \gg 10^{130}$ possible "texts" that you, a humble 21st-century human being, *can* (?) write in sequences s of 10^{10} bits on the hard disc of your tiny computer. Can't you?

How comes it that only a negligible percentage, less than $\frac{1}{10^{10^9}}$ of possibilities, can be actualized?

Worse than that, it is *impossible* to pinpoint a *single instance* of a non-realizable sequence s: Indicating an s will make this very s actual.

It is far from clear whether such inconsistency between "can" and "will" admits a clean mathematical reformulation or whether this belongs with the *paradox of the heap.* Yet, there are a few purely mathematical theorems and open problems that address this issue, albeit not satisfactorily.[163]

Is it Mathematics?

(1) It seems not hard to show that there exist lots of *provable* theorems T that can be formulated on a page or two in the standard mathematical language but such that their shortest proofs must be enormously long; hence, *humanly unprovable.* (To be unprovable one does not need "enormous" – "modest" 100^{100} will do.)
(2) On the other hand, pinpointing a *specific* a priori provable T that is *realistically unprovable* may be humanly impossible.
(3) A somewhat more problematic (and more interesting) possibility is that there exists a theorem T' that admits a hundred page proof but a search for the proof of T' would need at least 10^{50} human + computer hours; hence, being unfindable.

Explanation. Unlike (1) and possibly (2), that (almost) entirely belong with mathematics proper, this (3) hardly can be approached without making some conjectures on the innate resources of the human (ergo)mind/brain.

In fact, all kind of arguments and ideas – computations, proofs, etc. – that we can imagine and/or design, let this be done with a use of computers, are limited

[162]Archimedes, recall, evaluated the number of sand grains that would fill the Universe as $\approx 10^{60}$ where *exponential representation of numbers* was invented by him for this purpose.
[163]These turn around the following question. What makes it so difficult to locate/construct *individual* objects O with a property P from a given class C of similar objects, when we know that this P is satisfied by the *majority* of members of C?

to *compositions* of what is available in the "pool" of a *relatively small number* of innate atomic ideas evolutionarily installed by Nature into our (ergo)brain along with a collection of composition/selection rules of these ideas.[164]

Therefore, there exist "ideas" expressible in strings of 10^5 decimal symbols, that will NEVER be articulated in any form by humans on Earth.[165]

Shadows of Examples. By their very nature, *humanly inaccessible ideas* cannot be exemplified. But the following is suggestive.

($\star$) *"Colorless green ideas sleep furiously"* was probably meant by Chomsky as a *random grammatical*, hence, absurd sentence. But this is, in truth, a bona fide (sarcastic) English sentence due to pronounced, let them be negative, correlations between the words in it.[166]

(There is no 100% *random* option within your ergo-brain, but, given a list of words in front of you, you can (?) put your finger blindly on one of them.

NONSENSICAL SEVEN THUS CHOSEN ON RANDOM[167]

Illegal silly inflations mislead recklessly.
Extended meagre materials emphasize fortunately.
Fine joyous departments choose physically.
Wet coordinated articles complain coherently.
Wooden illustrious faults cost halfheartedly.
Wooden coordinated departments emphasize recklessly.
Illegal meagre departments complain halfheartedly.[168])

($\star\star$) To scientifically evaluate the intelligence of a monkey, a psychology professor suspended a banana high up under the ceiling, such that it could be reached only from a chair that had to be put on a table.

But the monkey, upon entering the room, discovered a better use of the professor's head than the professor himself: The animal jumped on the shoulder of the man, leaped up for the fruit using the man's head as a trampoline and safely landed on the table.

Apparently, this solution to the problem was absent from the toolbox of ideas under professor's skull.

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[164] Some of these "compositions" have already been implemented in the course of human cultural history with incorporation of bits and pieces from the "information flows" from the "external world", but this does not change the essence of what we say.

[165] If – which is unlikely – the number of equivalence classes formed by "meanings" carried by these strings is (very much!) significantly less than 10^{10^5}, then this is objectionable.

[166] Possibly, this sentence came up looking meaningful because it is too short.

[167] These are taken from the pool of, what I guess, 5 000–10 000 most common words out of "full" English vocabulary of more than 150 000 words.

[168] Notice, that the last two sentences are the diagonals of the first five and observe that the longer you look at these sentences the better sense they make – the human brain is apt to detect non-existent meaning in all kind of nonsense.

21. Probability: Particles, Symmetries, Languages

Can one reconcile Maxwell's thesis that

> *The true logic of this world is the calculus of probabilities.*

with Naum Chomsky's assertion that

> *The notion of a probability of a sentence is an entirely useless one, under any interpretation of this term*?

Human languages carry imprints of the mathematical structure(s) of the ergobrain and, at the same time, learning a natural (and also a mathematical[169]) language is a basic instance of the universal learning process of the human ergobrain. We can hardly understand how this process works unless we have a fair idea of what LANGUAGE is. But it is hard to make a definition that would catch the *mathematical essence* of the idea of LANGUAGE.

But isn't a language, from a mathematical point of view, *just*

a set of strings of symbols from a given alphabet,

or, more generally,

a probability distribution on the set of such strings?

A linguist would dismiss such definitions with disgust, but if you are a mathematician these *effortlessly* come to your mind. Paradoxically, this is why we would rather *reject* than accept them:

> *Mathematics is shaped by definitions of its fundamental concepts, but there is no recipe for making "true definitions." These do not come to one's mind easily, nor are they accepted by everybody readily.*

A good definition must tell you *the truth, all the truth, and nothing but the truth*, but there is no common agreement, not even among mathematicians, about what constitutes a TRUE DEFINITION.

For example, the idea of an *algebraic curve* that is a *geometric* representation of

solutions of a polynomial equation $P(x_1, x_2) = 0$ *in the* (x_1, x_2)*-plane*,
e.g., of the equation $2x_1^2 + 4x_2^4 + x_1x_2^3 = 0$,

by something like , originated in the work of Fermat and Descartes in the 1630s, and these curves have been studied in depth by generation after generation of mathematicians ever since.

But what is now seen as the simplest and the most natural definition of such a curve – the one suggested by Alexander Grothendieck in 1950s in the language of *schemes* – would appear absurd to anybody a few decades earlier.

Defining "language" and/or "learning" is, non-surprisingly, more difficult than "algebraic curve", since the former have non-mathematical as well as purely

[169] *Mathematical language* for us is the language used for communication between mathematicians but not a mathematical language of formal logic.

mathematical sides to them. They are similar in this respect to the concept of *probability*, that by now is a well-established mathematical notion.

It is instructive to see how "*random*" crystallized to "*probability*", and what was gained and what was lost in the course of this "crystallization".

Also, we want to understand how much of "random" in languages in the (ergo)learning process (including learning languages) is amenable to what Maxwell calls "the calculus of probabilities".

The concept of *chance* is centuries old as is witnessed by some passages in Aristotle (384–322 BCE) and also in the Talmud.[170] And Titus Lucretius (99–55 BCE), a follower of Democritus, describes in his poem *De Rerum Natura* what is now called the *Einstein–Smoluchowski stochastic model* of Brownian motion[171].

But mathematics of "random" was originally linked to gambling rather than to science.

I of dice possess the science and in numbers thus am skilled

said Rituparna, a king of Ayodhya, after estimating the number of leaves on a tree upon examining a single twig. (This is from *Mahabharata*, about 5 000 years ago; also 5 000 year-old dice were excavated at an archeological site in Iran.)

What attracts a mathematician to random dice tossing and what attracts a gambler are the two complementary facets of *stochastic symmetry*.

Randomness *unravels and enhances* the *cubical symmetry* of dice (there are $3! \times 2^3 = 48$ symmetries/rotations of a cube) – this is what fascinates a mathematician.

But randomness also *breaks* symmetries: The only way for a donkey' ergo-brain (and ours as well) to solve Bouridan's ass problem is to go random.[172] Emanation of the "miraculous decision power of random" intoxicates a gambler's ergo.[173]

The first (?) documented instance of the *calculus* of probabilities – "*measuring chance*" by a European[174] appears in a poem by Richard de Fournival (1200–1250)

[170]Our sketchy outline of the history of probability relies on [23] [9], [14], [16], [32], [27] with additional *References for Chronology of Probabilists and Statisticians* on Ming-Ying Leung's page, http://www.math.utep.edu/Faculty/mleung/mylprisem.htm

[171]This is the collective random movements of particles suspended in a liquid or a gas that should be rightly called *Ingenhousz' motion*.

[172]No deterministic algorithm can select one of two points in an (empty) 3-space as it follows from the existence of the *Möbius strip*. And a general purpose robot that you can ask, for instance, *bring me **a** chair* (regardless of several available chairs being identical or not) needs a "seed of randomness" in its software.

[173]In the same spirit, the ABSOLUTE ASYMMETRY of an individual random ± sequence of outcomes of coin tosses complements the ENORMOUS SYMMETRY of the whole space S of dyadic sequences that is acted upon by the compact Abelian group $\{-1,1\}^{\mathbb{N}}$ for $\mathbb{N} = \{1,2,3,4,5,\dots\}$ and by automorphisms of this group.

[174]Some "calculus of probabilities", can be, apparently, found in the *I Ching* written about 31 centuries ago.

who lists the *numbers* of ways three dice can fall. (The symmetry group in the case of n dice has cardinality $n! \times (48)^n$; that is, 664 552 for $n = 3$.)

Next, in a manuscript dated around 1400, an unknown author correctly solves an instance of *the problem of points*, i.e., of division of the stakes.

In 1494, the first (?) treatment of the problem of points appears in *print*[175] in Luca Paccioli's *Summa de Arithmetica, Geometria, Proportioni et Proportionalità*.[176]

Paccioli's solution was criticized/analized by Cardano in *Practica arithmetice et mensurandi singularis* of 1539 and later on by Tartaglia in *Trattato generale di numerie misure*, 1556.

Cardano

Gerolamo Cardano was after Vesalius the second most famous doctor in Europe. He suggested methods for teaching deaf-mutes and blind people, a treatment of syphilis and typhus fever. Besides, he contributed to mathematics, mechanics, hydrodynamics, and geology. He wrote two encyclopaedias of natural science, invented the *Cardan shaft* used in today's cars, and published a foundational book on algebra. He also wrote on gambling, philosophy, religion, and music.

The first (?) systematic mathematical treatment of statistics in gambling appears in Cardano's *Liber de Ludo Aleae*, where he also discusses the psychology of gambling, written in the mid 1500s, and published in 1663.

In a short treatise written between 1613 and 1623, Galileo, on somebody's request, effortlessly explains why upon tossing three dice, the numbers (slightly) more often add up to 10 than to 9. Indeed, both

$$9 \underset{1}{=} 1+2+6 \underset{2}{=} 1+3+5 \underset{3}{=} 1+4+4 \underset{4}{=} 2+2+5 \underset{5}{=} 2+3+4 \underset{6}{=} 3+3+3$$

and

$$10 \overset{1}{=} 1+3+6 \overset{2}{=} 1+4+5 \overset{3}{=} 2+2+6 \overset{4}{=} 2+3+5 \overset{5}{=} 2+4+4 \overset{6}{=} 3+3+4$$

have six decompositions, but $10 = 3+3+4 = 3+4+3 = 4+3+3$ is thrice as likely as $9 = 3+3+3$.

(If you smile at the naivety of people who had difficulties in solving such an elementary problem, answer, instantaneously,

> *What is the probability of having two girls in a family with two children where one of them is known to be a girl?*[177])

The formulation of basic probabilistic concepts is usually attributed to Pascal and Fermat who discussed gambling problems in a few letters (1653–1654) and to Huygens who in his 1657 book *De Ratiociniis in Ludo Aleae* introduced the idea of *mathematical expectation.*

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[175] The first book printed with movable metal type was the Gutenberg Bible of 1455.

[176] Paccioli became famous for the system of *double-entry bookkeeping* described in this book.

[177] This would take half a second for Galileo – the answer is 1/3 ($\pm\varepsilon$).

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But the key result – *the Law of Large Numbers* (hinted at by Cardano) was proved by Jacob Bernoulli only in 1713.

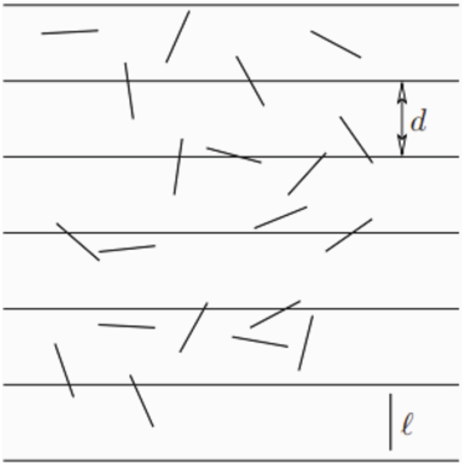
This, along with the *Pythagorean theorem* and the *quadratic reciprocity law*[178] stands among the ten (±2) greatest mathematical theorems of all time.

"*Continuous probability*" was invented in 1733 by Buffon who thought of

a needle of unit length (*instead of dice*) *randomly thrown on the plane, where this plane was divided into parallel strips of unit width.*

He proved that

the probability of crossing a line between two strips by the needle equals $2/\pi$ *for* $\pi = 3.14\ldots$ *being one half the length of the unit circle.*



Das BUCH der Beweise, Das Nadel-Problem von Buffon, 2015, Seite 199, Martin Aigner & Günter M. Ziegler (Springer-Verlag Berlin Heidelberg 2015)

[178]Let p, q be odd primes and $q^* = (-1)^{(q-1)/2}q$. Then $n^2 - p$ is divisible by q for *some* integer n if and only if $m^2 - q^*$ is divisible by p for *some* m.

Georges-Louis Leclerc Buffon

Besides opening the fields of *geometric probability* and *integral geometry*, Buffon, who understood the physics of his time, contributed to theoretical and practical optics. As an experiment, he constructed a large (about 2 m) concave mirror built of 360 small ones that, by focusing sunlight, could melt iron at 10 m distance and ignite wood at 50 m.

But his major contribution, about which we wrote in Chapter 6 of Part I, was to what he called "natural history" – a development of a synthetic picture of Life on Earth, where he outlined many essential interactions between organisms and their environment, much of which now goes under the heading of "biogeography".

Buffon emphasized the preeminence of biological reproduction barriers between different groups of organisms over the obvious geographical ones that suggested a definition of *species* that has withstood the attempts to "improve" it by later natural philosophers including some 20th-century post Darwinian evolutionary thinkers.

Buffon was the first (?) who articulated the main premise of evolutionary biology – the concept of the *common ancestor of all animals*, including humans.

Buffon's view on Nature and Life, expounded in his *Histoire naturelle, générale et particulière* published between 1749 and 1789 in 36 volumes, became a common way of thinking among educated people in Europe for two centuries afterwards.

With the Buffon's needle, "random" merged with "analysis of continuum" and were empowered by "calculus of infinitesimals". This is what was hailed by Maxwell and exploited by generations of mathematicians and physicists after Buffon.

This calculus comes at a price: Probability is a "full fledged number" with the addition/multiplication table behind it. But assigning a *precise specific* numerical value of probability to a "random event" in "real life", e.g., to a sentence in a language, is not always possible.

On Symmetry in Randomness

The elegance and success of probabilistic models in mathematics and science (always?) depends on (often tacitly assumed and/or hidden) symmetry.

The essentiality of "equiprobable" was emphasized by Cardano and *parametrization* of random systems by "independent variables" has always been the main tenet of probability theory. Most (all?) of classical mathematical probability theory was grounded on (*quasi*)*invariant Haar*(*-like*) *measures* and the year 2000 was landmarked by the most recent triumph of "symmetric probability" – the discovery of (essentially) *conformally invariant* probability measures in spaces of planar curves (and curves in Riemann surfaces) parametrized by increments of *Brownian's processes* via the *Schram–Loewner evolution equation.*

Atoms and Bacteria. A bacterium-sized speck of matter may contain, say, $N_{AT} = 10^{12}$ atoms and/or small molecules in it, and the number N_{BA} of bacteria residing in your colon is also of order 10^{12}. If there are two possible states for everyone – be they atoms or bacteria – then the number of *conceivable* states of the entire system, call it S, is the monstrous

$$M = M(S) \geq 2^{10^{12}} > 10^{3\,000\,000\,000}$$

where its reciprocal,

$$\frac{1}{M} < 0.\underbrace{000\ldots 0001}_{3\,000\,000\,000}$$

taken for the probability of S being in a particular state, is too small for making any experimental/physical/biological sense.

However, the assignment of the $\frac{1}{M}$-probabilities to the states is justified and will lead to meaningful results IF there is a symmetry that makes these tiny meaningless states "probabilistically equivalent", where the nature of such a symmetry, if it is present at all, will be vastly different in physics and in biology.[179]

But if there is not enough symmetry and one cannot *postulate* equiprobability (and/or something of this kind such as *independence*) of certain "events", then the advance of classical calculus stalls, be it mathematics, physics, biology, linguistic, or gambling.

On Randomness in Languages

Neither unrealistic smallness of probabilities, nor failure of "calculus with numbers", preclude a use of probability in the study of languages and of learning processes. And if you are too timid to contradict Chomsky, just read his

> *under any interpretation of this term* as
> *under any interpretation of the term probability*
> *you can find in a 20th-century textbook.*

Probability applicable to languages must not be an assignment of *numbers* to events, but rather a "*functor*" from a "linguistic category" to a "small and simple" category, yet, in general, different from the "category of numbers".

And forfeiting numbers for probabilities in languages is unsurprising – numbers are not the primary objects in the ergo-world. Numbers are not there, but there is a visibly present *partial order* on "plausibilities" of different sentences in a language. This may not look like much, but a *hierarchical use* of this order allows recovery of many linguistic structures.

[179] It is not fully accidental that the numbers N_{AT} and N_{BA} are of the same order of magnitude. If atoms were much smaller or cells much bigger, e.g., if no functional cell with less than 10^{20} atoms (something slightly smaller than a Drosophila fly) were possible, then, most probably, LIFE, as we know it, could not have evolved in our short-lived Universe with hardly 10^{80} atoms in it.

Adaptation of probability to the needs of (ergo)linguistics will also require a revision of the concept of "event" the probability of which one measures.

The nowadays canonized definition of "event", suggested in 1933 by Kolmogorov in his *Grundbegriffe der Wahrscheinlichkeitsrechnung*, is essentially as follows.

> *Any kind of randomness in the world can be represented* (*modeled*) *geometrically by a subdomain Y in the unit square* ■ *in the plane. You drop points to* ■, *you count hitting Y for an* ***event****, and define the probability of this event as area*(Y).

However elegant this *set-theoretic* frame is (with ■ standing for a *universal probability measure space*), it must share the faith of André Weil's *universal domains* from his 1946 book *Foundations of Algebraic Geometry*. The set-theoretic language introduced in mathematics by Georg Cantor that has wonderfully served us for almost 150 years is now being supplanted by a more versatile language of *categories* and *functors*. André Weil's varieties were superseded by Grothendieck's schemes, and Kolmogorov's definition will eventually go through a similar metamorphosis.

A particular path to follow is suggested by Boltzmann's way of thinking about statistical mechanics – his ideas invite a use of *non-standard analysis* as well as of a Grothendieck's style *category theoretic* language. But a mathematical interpretation of the idea of probability in *languages* and in *learning* needs a more radical deviation from (modification? generalization of?) this ■.

Cardano, Galileo, Buffon. The very existence of these people challenges our vision on the range and extent of the human spirit. There is no apparent wall between the ergos and egos in the minds of these men.

Where are such people today? Why don't we see them anymore? Nobody in the last 200 years had a fraction of Cardano's intellectual intensity combined with his superlative survival instinct. Nobody since Buffon has made long-lasting contributions to domains as far-distant one from another as pure mathematics and life sciences. What needs to be done to bring Galileos back to us?

22. Signal Flows from the World to the Brain

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The signals entering the ergo-brain via vision, hearing and olfaction are "written" on certain physical/chemical backgrounds, the structures and symmetries of which are mathematically quite transparent.

1. The visual signals are carried by the four-dimensional *space* + *time* continuum. Signals break the symmetry (and continuity) of the *space* + *time* but eventually, the ergo-brain reconstructs this symmetry.
2. The auditory signals are carried by the three-dimensional *time* + *frequency* + *amplitude* space. Ergo-brain, unlike mathematicians, does not seem to care about the underlying (symplectic) symmetry of this space; it is concerned with the "information content" of these signals and with *correlations* and/or *redundancies* in flows of the signals.
3. Structurally, olfaction has much in common with gustation as well as with perception of warm, cold, and pain, that are, however, more primitive than olfaction in most animals. But olfaction is fundamentally different from vision and hearing.

Scents, unlike images and sounds, cannot be written and rewritten on a variety of backgrounds they are supported by physical/chemical substrates *specific to each of them.*

The "letters of smells" are aromatic molecules entering our noses. The flows of these, *unless they are bound to a rigid substrate*,[180] have more internal symmetries than the flows of visual and auditory signals: N different kinds of aromatic molecules that freely float in a gas or in a liquid can be permuted in $N! = 1 \times 2 \times 3 \times \cdots \times N$ different ways.[181]

No rewriting, no natural digitalization, no universal structuralization of smells comparable to that of images or sounds is possible.

[180] Scents attached to a rigid surface, allow, for instance, tracking by dogs.
[181] This is, of course, an exaggeration. Nothing on Earth has ever been permuted in $50! > 10^{64}$ ways.

Because of this smells cannot be positionally codified and olfaction inputs have significantly lower information contents than better organized flows of visual and auditory signals. (This is an apparent reason for why the olfactory perception depends on so many *different*[182] receptors.)

For instance, combinations of two different kinds of molecules, ⬡ and □, can encode only a few dozen distinct signals, because there is no perceptible order in how groups of these molecules, say from the following string, may enter your nose.

⬡□⬡⬡□⬡⬡□□□⬡□⬡□□□⬡□□⬡⬡□⬡□□□□⬡⬡□

Even though there are 2^{30} – more than a billion – of such binary strings, your nose can only perceive the relative amount of ⬡ and □. (This is a humble 13/15 for the ⬡□-string.)[183]

Possibly, bloodhounds, who have 200–300 million olfactory receptor neurons in their nasal cavities, may differentiate smells from a pool of size 10^6(?), 10^9(?), 10^{12}(?)[184], but they cannot beat a human visual system that can easily distinguish randomly-taken images from a pool of 10^{20} and with a little effort (training?)[185] up to 10^{1000}.

The internal library of smells is simpler than how we remember visual images, sounds, words, and ideas – no ergo is needed for organisation of this "library".

Besides, olfaction, unlike vision, does not depend on your muscles, it is disconnected from the proprioception system and we have no means of (re)producing scents at will, albeit we think we can recall them.

There aren't many clearly identifiable *universal smells* common to large groups of object. Non-surprisingly, languages (of urban populations?) have few specific names for smells – about ten in English:

musky, putrid, rotten, floral, fruity,
citrus, vegetative, woody, herbaceous, spicy.

(There are slightly more smell-words in certain languages, for example there are about fifteen of them in the *Kapsiki of Cameroon*.)

Natural languages do not waste words for naming *individual* objects/properties but rather exercise the art of giving the same name to *many* different things, very much as mathematical theories do. There is no grammar of scents, no books in the language of odors, no possibility to encode and "freeze" the flows of olfactory

[182] There are over 1000 different receptor proteins in these neurons, as was established by Richard Axel and Linda Buck.

[183] Variation of smell intensity in time does not contribute much, although this may work as a clock for a dog.

[184] Apparently, there is neither an unequivocal definition of "olfaction sensitivity" nor reliable data for a convincing estimation of the olfaction capacity of people and other animals.

[185] Strings of common words are easy to discriminate but see how long it takes to distinguish the following two.

अ ल ो ए व अ न अ गअ रई हअ तए ह अइ and अ ल ो ए व अ न अ अग रई हअ तए ह अइ.

perception.[186] (This is what stopped bloodhounds from developing a language based on olfaction.)

4. There is no apparent uniform (symmetric) spacial background for somatosensory and touch (haptic) perceptions but their information carrier potential is comparable to that of vision and hearing.

Seven Flows

... think of some step that flows into the next one, and the whole dance must have an integrated pattern.

Fred Astaire

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Incoming flows of signals are naturally classified according to the sensory receptors and pathways by which they enter the brain: *Visual, auditory*, and *somatosensory*, where the two relevant aspect of the latter are *proprioception*, the body sense, and *tactile*, i.e., *touch perception.*

But from an ergo-learner perspective, signals differ by how one learns their "meanings", how one interacts with them, how one arrives at an understanding of their structures.

1. *Spoken language* depends on the auditory and sensory-motor systems; ears to listen and sensory-motor systems to generate speech. However, deaf-mute people speak in sign language and deaf blind people communicate in tactile sign language.[187]

2. *Written language* (whenever it naturally exists) is likely to have a huge overlap with the spoken one in the human brain (of a habitual reader) but it also makes a world of its own. It is not inherently interactive, at least not so superficially,[188] and it is not *physically* bound to the flow of time.[189] Persistence of written literature is hard to reconcile with a naive selectionist view on co-evolution of language and the brain.

3. *Mathematics.* Learning mathematics is an interactive process but it is hard to say exactly in what sense.

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[186] Perfumes do not count.

[187] Most amazingly, some deaf blind people can understand spoken language by picking up the vibrations of the speaker's throat.

[188] Writing and reading is kind of talking to oneself.

[189] The time arrow is implemented by *directionality* of what is written.

The images a mathematician generates in his/her mind are neither of language nor do they belong with any particular "sensory department". Thinking mathematics is like driving an imaginary bicycle or performing/designing a dance with elaborate movements entirely in your head. (This may differ from person to person.)

4. *Languages of games.* We are able to enjoy and to learn a variety of mental and physical games. Probably, these are divided into several (about dozen?) classes depending on how they are incorporated into our ergo-brains. Written language and mathematics may be particular classes of games.

5. *Music.* People gifted in music replay melodies in their minds and they can reproduce melodies vocally and/or with musical instruments; the rare few may generate new melodies. But melodies, unlike sentences in a language, cannot talk about themselves and there is no general context in which one can formulate what human (unlike that of birds) music is and/or what should be regarded as "understanding music".[190]

6. *Proprioceptive/somatosensory system.* Running over a rough unpredictable terrain is kind of talking to the road with the muscles in your body. This is much simpler than ordinary language but is still beyond the ability of computers that control robots. Neither is a present day robot able to sew a button on your shirt.

7. *Vision.* At least half of the neocortex in humans is dedicated to vision, but this may be mainly due to the sheer volume of the information that is being processed and stored, rather than to the structural depth of visual images.

© Crystal Eye Studio / Shutterstock.com

[190]Recently, there was an attempt to understand what music does to one's brain: http://phenomena.nationalgeographic.com/2013/04/11/why-does-music-feel-so-good/ and http://www.zlab.mcgill.ca/home.php?1592876871.

This is probbaly, why vision impairment, even vision + hearing impairment, do not significantly affect human ergo. The ergo is robust and independent of particular sensory inputs as exemplified by people such as Hellen Keller.

Three flows among these, *Language, Mathematics, and Music*, have an essential feature in common: The receiver of such a flow $\overrightarrow{F}$ develops an ability, with no external reinforcement, to creatively generate a new flow $\overleftarrow{F}$ in the class of $\overrightarrow{F}$. (In the case of Mathematics and Music this happens rarely, but the miracles of this having happened in the brains of Mozart and Ramanujan outweighs any statistics.)

Modelling the transformation $\overrightarrow{F} \mapsto \overleftarrow{F}$ is one of the key aspects in our picture of the universal learning problem. (Possibly, there are counterparts of $\overleftarrow{F}$ for other incoming flows $\overrightarrow{F}$, but they may be kind of *internal*.)

The most studied among these is the learning of native languages by children, the (still unknown) mechanism of which must not be not far from how mathematics is learned by mathematicians.

The structure of a most sophisticated mathematics we build in our minds is likely to be simpler than that of natural languages (and smaller than that of vision), but it is still quite interesting, and the corresponding learning process may be more accessible, due, besides its relative simplicity, to a great variance in people's abilities in learning mathematics[191] and a presence of criteria for assessing its understanding.

Transformations and Reconstruction of Flows: Learning to Read by Learning to Speak

The original form of signals carried by the above seven flows is different from what arrives at your sensory systems. For instance, visual images result from 2D *projections* of *three-dimensional* patterns to the retina in your eyes; moreover, the brain's analysis of these projections is coupled with the activity of the brain's motor system that controls movements of the eyes that continuously modify these projections.

Similarly, the flow of speech as it is being generated in one's mind has, according to the tenets of *generative grammar*, a tree-like structure that is then "packed" into single time line.

Reconstruction of F_{orig} from the flow F_{rec} that you receive is an essential aspect of understanding the message carried by F_{orig}. For example, understanding a flow of speech is coupled with one's ability to speak, i.e., to reconstruct/generate F_{orig}, or something close to it, in one's ergo-brain.

This reconstruction can be expressed, albeit incompletely, as *annotation* to F_{rec}.

[191] Every sane person understands his/her mother tongue and has an adequate visual picture of the world. This uniformity makes understanding of these "understandings" as difficult as would be understanding motion in a world where all objects moved in the same way.

For instance, upon receiving a flat image on its screen (retina), an ergo-learner $\mathcal{L}$ must correctly resolve depth in *interpositions/occlusions* and/or "guess" relative values of the third coordinates at essential points of this image.

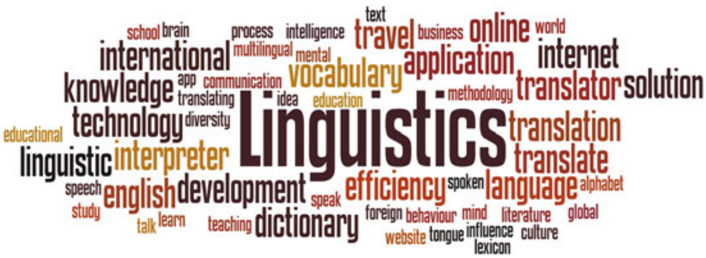
And the background tree structure in a (record of a) flow of speech can be indicated with *parentheses* properly inserted into sentences. (An annotation may also include additional syntactic and/or semantic comments concerning particular words and sentences.)

Such annotations performed by a human ergo-brain depend on an elaborate guesswork that is by no means simple or automatic. And besides annotating flows of signals, the ergo-brain augments them by something else.

For instance, the formation of a visual image in one's mind depends on the activity of motor neurons involved in eye movements, and "understanding" of these images depends on structural matching of this activity with similar actions of these neurons in the past.

This active process of perception can be seen as a conversation or a kind of a game of the ergo-brain with the environment. But such games, unlike anything like chess, are not easy to mathematically formalize.

23. Characteristic Features of Linguistic Signals



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Linguistic information entering the ergo-brain, unlike chemical messaging between cells, does not much depend on the physical carrier of this information and on the corresponding perceptual channels, be these auditory, visual (in reading and in sign languages), or tactile (in deaf-blind communication).[192] The properties of "verbal signals" reflect the properties of the underlying *syntactic ergo-structure*, the apparent attributes of which are as follows.[193]

(1) *Fast Language Specific Clustering.* Formally/physically *different* signals, e.g., sounds, are perceived/recognised as *identical* verbal units, e.g., phonemes, words, phrases, where this is achieved within *half-second* time intervals.

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[192]Languages for a physicist are instances of flows of (very!) *low energy* signals that are able, however, to *strongly perturb* the dynamics of otherwise stable systems, such as the physiology of the human body.

[193]We do not speak here of *linguistic universals* such as *negative*, *interrogative*, and *antonymous* that are absent from non-linguistic signals.

The clusterization of *phonemes* (and, probably, of other, including non-auditory, basic verbal units) depends on a particular language, and the mechanism of learning these clusters by children (that deteriorates with age) is poorly (if at all) understood.

Yet, abstractly speaking, this is the easiest of our problems as is witnessed by the efficiency of (non-contextual?) speech recognition algorithms.

(2) *Formalized Division into Units.* Flows of speech are *systematically* divided (albeit non-perfectly) into (*semi*) *autonomous units*, where the basic ones are what we call *"words"*.

This division, that is sharper than that of signals coming from "natural sources", is based to a significant extent on *universal* principles of *segmentation* that are applicable to all kind of signals, where the markers separating "segments" are associated with *pronounced minima* of the *stochastic prediction profiles* of signal flows, where determination of such a profile depends on structural patterns characteristic for a particular flow.

(3) *Medium and Long Range Structure Correlations.* There are more "levels of structure" in languages than in other flows of signals. This is seen, in part, in a presence of non-local "correlations" between different fragments in texts.

For instance, if a sentence starts with "*There are more ... in ...*" one may rightly expect "*than in*" coming next with abnormally high probability.[194]

And if "*Jack*" appears on every second page in a book and "*his eyes sparkled again*" than, you can bet, "*Jack's eyes sparkled*" appeared on the previous page.[195]

(4) *Verbal Reduction of non-Linguistic Signals.* Many different non-verbal signals, corresponding to objects, events, features, or actions may be encoded by the same word.[196] For instance, hundreds of small furry felines that have ever crossed your field of vision reduce to a single "cat".

Non-verbal signals are many while their word-names are few. The use of a language replaces the bulk of the "raw memory" in the brain by a network of "*understand*" links between individual items in this memory. This is why small children visibly enjoy the process of the verbal classification/unification of "natural signals" from the "external world" as they learn to identically name different objects.[197]

(5) *Imitation, Repetition, and Generation of Linguistic Signals.* Humans, especially children, have the ability to reproduce linguistic signals [*sign*] they receive, including those emitted by themselves, where, to be exact, *not signals* [*sign*] *themselves* are generated but members of the *same class/cluster* as [*sign*] and where the choice of a particular *classification rule* is not a trivial matter.

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[194] Try: *there are more * in* on Google.

[195] A universal learning program faces here the difficulty of "understanding" *his* and *again*.

[196] This may be contrasted with the existence of *synonymous* words, but the multiplicities and significance of the latter are incomparable to the power of verbal reduction.

[197] Children of this age are close to being ideal ergo-learners – striving to learn and to understand is the main drive of ergo-systems.

One can hardly analyse languages without being able to generate them,[198] where the language generative mechanisms – called *generative grammars* – result from the repetitive nature of imprecise imitation.

(6) *Many Levels of Self-Referentiality.* No other flow of signals, and/or human medium of communications have the propensity of self-reference typical of language. The ergo-structures of languages contain multiple reflections of their own "selves", their internals "egos", such as

noun-pronoun pairs, allusions to previously said/written items, summaries of texts, titles of books, tables of contents, etc.

Understanding a language is unthinkable without the ability of generation and interpretation of self-referential patterns in this language.

(7) *Pervasive Usage of Metaphors.* Metaphors you find in dictionaries are kind of frozen reflections of their precursors in multiple colored mirrors of language (where such a precursor may have been long forgotten) that correspond to similar mirrors in human ergo-brains.

(Such "mirrors" in vision are, probably, implemented by projections from "deeper" regions of the brain to the primary visual cortex.)

Language in the ergo-brain

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Mental representations of languages are described by *syntactic/linguistic ergo-structures* that are similar, but are in a way more *elementary and more abstract* than what is customarily studied by mathematicians and by linguists. (*Elementary, abstract, fundamental, rudimentary* are synonymous from the ergo-point of view.)

We do not have a full idea of what these are but the *combinatorial core* of such a structure is visible as a *pair of multilayered colored networks*:

The first network is comprised of connectives between (syntactically or semantically) interacting (often spatially/temporally close) units in texts (conversations) and the second one is composed of various kinds of similarities between such units.

And also (non-group theoretic) symmetries implied by transformation rules within these structures bring to mind what one encounters in the *theory of n-categories.*

[198] Neuronal signal generation mechanisms play an essential role also in vision: much of what you "really see" is conjured by your own brain, but the details of this process are inaccessible to us.

24. Understanding Structures and the Structure of Understanding

If there was a parrot which could answer every question,
I should say at once that it was a thinking being.

DIDEROT, PENSEES PHILOSOPHIQUES, 1746

But ...

> *It never happens that it* [an automaton] *arranges its speech in various ways, in order to reply appropriately to everything that may be said in its presence, as even the lowest type of man can do.*
>
> DESCARTES, DISCOURSE ON METHOD, 1637

Is Descartes justified in his belief that no machine can pass what is nowadays called *Turing Test*; i.e., *to reply appropriately to everything that may be said in its presence*?

Does passing such a test certify one as a THINKING BEING who UNDERSTANDS what is being said, as Diderot maintains?

What does it mean to UNDERSTAND, say, a language or any other flow of signals?

Diderot indicates a possible answer:

the continuity of ideas, the connection between propositions, and the links of the argument that one must judge if a creature thinks.

To go further, we conjecture that

- most (all?) structures we encounter in life, such as natural languages, mathematical theories, etc. are UNDERSTANDABLE;[199]
- this UNDERSTANDING universally applies to a large(?) class of *structural*, entities;
- UNDERSTANDING is an elaborate structural entity in its own right, thus, being subject to mathematical scrutiny.

Granted all this, we start searching for mathematical models[200] of UNDERSTANDING.[201]

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[199] Over-optimistic? Yet, in line with the remark "... *mystery of the world is its comprehensibility*" by Einstein.

[200] When we say *mathematical structure* or *mathematical model* we do not have in mind any particular branch of the continuously growing, and mostly hidden from us, enormous tree that is called MATHEMATICS.

[201] Impossibility of resolving – not even formulating – the problems of UNDERSTANDING and of *thinking machine* in simple words does not abate one's urge to make the world know what one's *gut feeling* about these issues is – an incessant flow of publicised opinions on this subject matter is a witness to this.

Amusingly, the *gut feeling* itself, at least the one residing in a dog's guts, unlike the ideas propagated from human guts to human minds, was experimentally substantiated by H. Florey with his coworkers, with the results published in 1929 under the title *The Vascular Reactions of*

Answers to the following questions, let these be only approximate ones, may serve to narrow the range of this search.

QUESTION 1. What are essential (expected? desired?) features/architectures of mathematical models of structural understanding?

QUESTION 2. If such a model exists should it be essentially unique? In particular, must the hypothetical structures of understanding, say of a language and of chess, necessarily closely resemble one another?

QUESTION 3. How elaborate does such a model need to be, and, accordingly, how long should be a computer program implementing such a model?

QUESTION 4. What is an expected time required for finding such a model and writing down the corresponding program?

QUESTION 5. What percentage of this time may be delegated to machine (ergo)-learning with a given level of supervision?

QUESTION 6. How much of the supervision of such learning can be automated?

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QUESTION 7. What are criteria/tests for performance of "I UNDERSTAND" programs?[202]

QUESTION 8. Can Turing-like tests be performed with *algorithmically* designed formal questions that would trick a computer program to give senseless answers?[203]

QUESTION 9. Are there simple rules for detecting senseless answers?

the Colonic Mucosa of the Dog to Fright. (This the same Florey who was responsible for bringing penicillin to the therapeutic use in 1940s.)

[202]A designer's own ability to pass a test is a poor criterion for designing such a test.

[203]Such questions must refer to an earlier part of a conversation. For instance, the examiner says "*... example ...*" at some point and then, responding to a question by the program, the examiner says; "*I have already given an example of this in my previous five word sentence.*" But to trick a sufficiently sophisticated program one must design a "logical *Russian doll*" of such questions.

QUESTION 10. Can the human learning (teaching?) experience be of use for designing clever learning algorithms?

QUESTION 11. Does ergo-logic help answer the above questions?

Apparently, UNDERSTANDING is composed of three ingredients:

$[\bullet]_U$ *a certain combinatorial (logical?)* ***structure*** $\mathcal{U}$ *in the understander's mind/brain/program;*

$[\bullet]_{[IU]}$ *a* ***process*** $\mathcal{IMPU}$ *of implementation of* $\mathcal{U}$ *by an ergo-system representing "an understander";*

$[\bullet]_{[RU]}$ *the* ***result*** $\mathcal{RESU}$ *of such implementation,* $\mathcal{RESU} = \mathcal{IMPU}(\mathcal{FLOW})$, *where* $\mathcal{IMPU}$ *is seen as a transformation applied to flows of signals.*

We do not know for sure if *understanding* is a formalizable concept and, if "yes", there is no clear idea of what kind of structure this $\mathcal{U}$ could be.

The only convincing argument in favour of the existence of $\mathcal{U}$ would be designing a functional *thinking machine/program*, and the only conceivable NO might come from an incredible discovery of a hitherto unknown fundamental property of the live matter of the brain.

This precludes any speculation on how and where such a structure can be implemented. Besides such a structure is by no means unique but rather different $\mathcal{U}$ are organized as a structural community that can be partly described in category-theoretic terms.

However, we have a realistic(?) expectation of what space/time characteristics of this structure(s) could be: $\mathcal{U}$ is much smaller in volume content than the totality of the flow $\mathcal{FLOW}$ this $\mathcal{U}$ "understands", and implementation of $\mathcal{U}$, that is application of $\mathcal{IMPU}$ to a flow $\mathcal{FOW}$, is much faster than achieving understanding $\mathcal{U}$.

It takes, probably, $\approx l \log l$ elementary steps for learning $\mathcal{FLOW}$ of length l that translates to months or years when it comes to learning a language or a mathematical theory.[204] But when learning is completed, it takes a few seconds to realize, for instance, that a certain string of symbols in the language of your $\mathcal{FLOW}$ is completely novel and even less time to see that a string is meaningless.

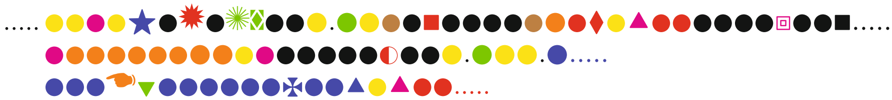
On the other hand, the space/volume occupied by an understanding program $\mathcal{U}$ is a few orders of magnitude greater than a learner's starting program $\mathcal{PROG}$, where such a program is *universally* (independently of the total number of signals from $\mathcal{FLOW}$ received/inspected by a learner) bounded by something like 10^6 bits. Picturesquely,

$$\overset{\star}{\mathcal{PROG}} \rightsquigarrow \underbrace{\circledast \equiv \blacksquare - \blacksquare - \blacksquare - \blacksquare - \blacksquare - \blacksquare - \blacksquare}_{\mathcal{U}}$$

where $\circledast$ represents the "core understanding" – a few thousand page "dictionary + grammar" of $\mathcal{FLOW}$ – the flow of signals, where this $\circledast$ is augmented by several

[204] The true measure of time, call it *ergo-time*, should be multi-(two?)-dimensional, since it must reflect parallelism in programs modelling learning and other mental processes.

(tens, hundreds, or thousands depending on $\mathcal{FLOW}$) “volumes” of loosely (imagine RAM on your commuter) organized “knowledge”, while the available $\mathcal{FLOW}$ itself may number in tens or even hundreds of millions of nearly unrelated units – volumes, internet pages, images – a mess like this:



The branch of mathematics that would support, say human UNDERSTANDING, of language, must be quite elaborated – it is not grown yet on the “grand tree of math.”

But sometimes things are simple, e.g., for the vervet monkey “*Alarm Call Language*” that matches a few (four?) word-signs – their alarm calls, with object-events, that are particular predators – leopards, eagles, pythons, baboons.

However, no monkey would think that a mathematical *one-to-one correspondence*, call it AlCaLa, between two 4-element sets *understands* the meaning of the alarm calls even if this AlCaLa is implemented by a monkey-shaped robot that properly reacts to predators by correctly emitting the corresponding calls and, thus, passes the vervet monkey Turing test.

Why then do Descartes and Diderot, not to speak of Turing himself, attach such significance to the Turing test?

Is there an *essential* difference between the correspondence “*questions*” $\longrightarrow$ “*correct answers*” in a human language and the AlCaLa correspondence?

The answer is:

Yes, there is an essential difference, an ENORMOUS DIFFERENCE.

Operating with tiny sets, e.g., composed of four utterings – alarm calls of vervet monkeys and with correspondences between such sets needs no structure in these sets. But one CANNOT manipulate human utterings and even less so longish strings of utterings and/or written texts in a structureless way.

It is tacitly assumed by scientifically minded people – Descartes, Diderot, Turing ... – that the above correspondence “$\longrightarrow$” must be compatible with the essential structure(s) of the human language, call it $\mathcal{HL}$, used in a particular Turing test, where the basic (but not the only) structure in $\mathcal{HL}$ is that of an *exponential/power set*:

An uttering/sentence, say in thirty words, in a language with dictionary D is seen as a member of the HUGE *power set*

$$D^{30} = \underbrace{D \times D \times D \times \cdots \times D}_{30}.$$

Such structurality is indispensable for an implementation of a “thinking automaton” and/or the program running it in a *realistic* space time model [205] that

[205]The property of being physically realistic is often missing in philosophical discourses on artificial intelligence.

necessarily excludes, for instance, "imaginary programs" containing in their memories lists of more than, say N^{15}, sentences with the number N being comparable with the cardinality $card(D)$ of the dictionary.[206]

("Large sets", be they finite or infinite, have no independent existence of their own, but only as carriers of structures in them, similarly to how the space-time in physics makes no sense without energy-matter in it. This is not reflected, however, in the set-theoretic notation that may mislead a novice. For instance, it is rarely stated in elementary textbooks that the "correspondence" $x \mapsto y$ in the "definition" of a *real variable function* $y = f(x)$ is only a metaphor and that a function $f(x)$, if it claims the right to exist in mathematics, must *"respect" some structure* in the set of real numbers.[207])

25. Sixteen Rules of an Ergo-Learner

The general guidelines/principles suggested by *ergo-logic* for designing universal learning algorithms can be summarized as follows.

1. Flows of signals coming from the external world carry certain structures "diluted" in them.

Learning is a process of extracting these structures and incorporating them into the learner's own *internal structure*.

2. The essential learning algorithms are *universal* and they indiscriminately apply to all kind of signals.[208]

3. Universality is incompatible with any a priori idea of "reality" – there is no mental picture of what we call "real world" in the "mind" of the learner.

The only *meaning* the learner assigns to "messages" coming from outside is what can be expressed in terms of (essentially combinatorial) *structures* that are recognized and/or constructed by the learner in the process of incorporating these "messages" in learner's internal structure.

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[206]This very sentence: "Such structurality is ... of the dictionary" contains forty words with roughly half of them being nouns, verbs, and adjectives. By varying these, one "can" generate more than $1000^{20} = 10^{60}$ grammatical sentences. Can one evaluate the number of meaningful ones among them? Would you expect *thousand* of them or, rather, something closer to *ten thousand*? Is it conceivable that "weakly meaningful" sentences number in 10^6, or are there more than 10^{10}, or even greater than 10^{18} of them?

[207]There is no accepted definition of "*function*" that would separate the *wheat*, $\sin x$, $\arctan x$, $\sqrt{x}$, $\log x$, Riemann's $\zeta(x)$, Dirac's $\delta(x)$, ..., from the *chaff*, such as the characteristic function of the subset of rational numbers.

[208]The learner's behaviour, that is learner's interaction/conversation with incoming signals, also depends on the learner's internal structure that has already been built at a given point in time. In particular, a prolonged exposure to a particular class of signals makes a learner's behaviour more specialized (more efficient?) but the learner's ability to absorb and digest different kinds of signals declines.

4. Universality also implies that the actions of the learner – building internal structures and generating signals, both within itself and/or released outside,[209] *are not governed by goals* expressible in terms of the external world.

The learning is driven by the learner's *"curiosity" and "interest"* in structural patterns the learner recognizes in the incoming flows of signals and in the learner's delight in the logical/combinatorial beauty of the structures the learner extracts from these flows and the structures the learner builds.

Essential ingredients of the learning process are as follows.

5. The learner discriminates between familiar signals and novelties and tries to match new signals with those recorded in its memory.

6. The learner tries to *structurally extrapolate* the signals already recorded in its memory in order *to predict* the signals that are expected to come.

7. Besides the signals coming from the external world, the learner perceives, records, and treats some *signals internally generated* by the learner itself.

8. The learner tends to *repeatedly imitate signals* being received, including some signals that come from within itself.

(The repetitiveness of their basic operations allows a description of learning processes as *orbits under some transformation in the space of internal structures of a learner*. The learning program that implements this transformation must be quite simple and the learning process must be robust. Eventually, "orbits of learning" stabilise as they approach approximately fixed points.)

9. The learner tends *to simplify signals* it tries to imitate.

10. The learner systematically makes guesses and "jumps to conclusions" by *making general rules* on the basis of regularities it sees in signals.

11. When the learner finds out that a rule is sometimes violated, the learner does not reject the rule but rather adds *an exception*.

12. The learner tends to use *statistically significant* signals for building its internal structure as well as for making predictions. But sometimes, the learner assigns significance to certain exceptionally rare signals and uses them as essential structural units within itself.[210]

13. The learner's probabilistic reasoning in uncertain environment is *yes-maybe-no* logic.

We impose the following restrictions on the abilities of our intended learner programs that are similar to those the human brain has.

14. The learner *does not accept unstructured sets* with more than four or five items in them; upon encountering such a set, the learner invariably assigns a certain structure to it.[211]

[209] These are the "actions" the human brain is engaged in.

[210] It is the rare words in texts that are significant, not the most frequent ones. For instance, *gifku mfink otnid* on three different pages of a book will impress you more than ooooo ooooo ooooo on twenty different pages.

[211] The partition of stars in the sky into constellations is an instance of this.

15. The learner has *no built-in ability of sequential counting* beyond 4 (maybe 3); we postulate that $5 = \infty$ for the learner.

In particular, the learner is not able to produce or perceive five consecutive iterations of the same process, unless this becomes a *routine* delegated from *cerebral cortex* to *spinal cord*.[212]

Our main conjecture is that

universal learning algorithms that converge to UNDERSTANDING *exist*
and that, moreover, their formalized descriptions are *quite simple*.

The time complexity of such algorithms must be at most log-linear (with no large constant attached) and the performance of an "educated/competent program" must be no worse than logarithmic.

In fact, the essential features of (ergo)learning as we know it, make sense only on a roughly "human" time/space scale: Such learning may apply to flows of signals that carry $10^9 - 10^{15}$ bits of information altogether and one hardly can go much beyond this.[213]

Universality and Doublethink. If one expects an analysis of *a flow of signals*, e.g., of a collection of texts in some language $\mathcal{L}$, to be anywhere close to the TRUTH, and if one wants to design an algorithm for learning $\mathcal{L}$, one must, following ergo-logic, *disregard* all one a priori knows about this $\mathcal{L}$, *forget* this is a language, and *reject* the idea of meaning associated to it.

But the only way to evaluate the soundness of your design, prior to a computer simulation of it, is to compare its performance to that of the corresponding algorithms in a human head.

[212] Never mind *the kid that fought his dad that bought the car that struck the bike that hit the truck that brought the horse that kicked the dog that chased the cat that caught the rat that ate the bread.*

[213] The universal learning systems themselves, e.g., those residing behind our skulls, have no built-in ideas of *meaning*, of *time*, of *space*, of *numbers*. But any speculation on natural or artificially designed "intelligent" systems strikes one as *meaningless*, if *spatial* and *temporal* parameters of possible implementations of such systems are *not specified* and set within realistic *numerical* bounds.

26. Learning to Understand Languages: From Libraries to Dictionaries

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Mathematically, the process of *learning a language, call it* $\mathcal{TONG}$ is represented by an *orbit* of the *universal learning program PROG* that acts on the *linguistic space of* $\mathcal{TONG}$, and where this orbit must eventually (approximately) converge to "*I understand* $\mathcal{TONG}$" state/program.

The principle of the existence of such a *PROG* is demonstrated by the linguistic performance of the brain of (almost) every child born on Earth that receives flows of *electro-chemical signals*, some of which come from linguistic sources, and the "meaning" of which the child's brain learns to "understand".[214]

A scenario closer to our experience is that of a visitor from another Universe[215] who attempts to "understand" what is written in some human "library" LIBR, e.g., on the English pages of the internet.

In either case, the process of what we call "understanding" is interpreted as assembling *an ergo-dictionary EDI* – a kind of "concentrated extract" of the combinatorial structure(s) that are present (but not immediately visible) in flows/arrays of linguistic signals.

Making a dictionary involves several *interlinked* tasks where a starting point is

Annotation & Parsing, that is identification and classification of ***textual units*** that are persistent and/or significant fragments in short strings s (say, up to 50–100 letter-signs) as well as attaching ***tags*** or *names* to some of these fragments.

Tagging may be visualized as coloring certain fragments in texts, where these fragments and the corresponding colors may overlap. Or, one may represent an

[214]Bridging linguistic signals to non non-linguistic ones is an essential but, probably, not indispensable ingredient of "understanding language" as it is suggested by the linguistic proficiency of deaf blind people.

[215]No imaginable Universe appears as dissimilar to ours as what the brain "sees" in the electro-chemical world where the brain lives.

annotation by several texts written in parallel with the original one, where the number of different color-words is supposed to be small, a few hundred (thousand?) at most, with a primitive "grammar" that is a combinatorial structure organizing them.[216]

One may think of such annotations as being written in strings positioned on several *levels*[217] over the original strings s, where the new tag-strings on the level l are written in the tag-words specific to this level and where the number of such l-tags (at least) decays exponentially fast with l.

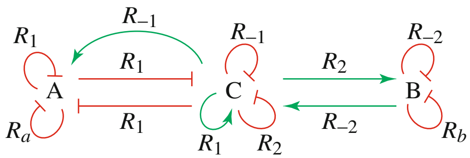
An ergo-dictionary is obtained by several consecutive *reductions* – kind of *compressions* – applied to a library of annotated texts where the resulting combinatorial structure of the dictionary is more elaborate than that of the library of annotated texts. The overall reduction in volume is a thousandfold, roughly, from 10^9–10^{10} to 10^6–10^7 linguistic units.

The grammar of a language makes a part of EDI where the structural position of this grammar in EDI is supposed to imitate how it is (conjecturally) organized in the human mind.

A particular ergo-dictionary

$$EDI = EDI(\mathrm{LIBR}) = EDI_{PROG}(\mathrm{LIBR})$$

is obtained from a collection of texts, what we call a *library* LIBR, according to some universal (functorial?) process/program $PROG$ that drastically reduces the size of LIBR and, at the same time, endows what remains with a *combinatorial structure* – a kind of "network of ideas" – that is similar to but more elaborate than the structure of a *partially directed edge and vertex colored graph.*



This EDI can be thought of as (a record of) *understanding* of the underlying language by the learner behind $PROG$. This understanding, call it U_t, is time-dependent, with EDI being an *approximate fixed point* of the learning process

$$U_{t_1} \underset{PROG}{\rightsquigarrow} U_{t_2},\ t_2 > t_1,$$

where, a priori, $PROG$ can be applied to "understandings" U that were not necessarily built by this $PROG$.

The essential problem here is finding a *uniform/universal* representation that can be implemented as a *coordinatisation* of "the space of understandings" U,

[216] An annotation may include references to *non-linguistic* signals but this would contribute to one's *knowledge* rather than to one's understanding.

[217] These levels l can be regarded as numbers $0, 1, 2, 3, \dots$ with $l = 0$ corresponding to strings in the original text, where the number of level is small, something between 3 and 5. But, as we shall see later on, these levels are organized according to a structure that is not quite a linear order.

where a simple minded program $PROG$ could act by consecutively adjusting "coordinates" $u_1, u_2, \ldots$ of U, and where this space would accommodate incoming loosely-structured flows of signals encoded by libraries as well as rigidly organized dictionary structures.[218]

The apparent ingredients of ergo-dictionaries EDI that encode "understanding" and processes for assembling these dictionaries are the following.

- *Short range correlations,*[219] *segmentation and identification/formation of units in flows of linguistic signals.*
- *Memory, information, and prediction on different levels of structure.*
- *Similarities, equalities, contextual classification, cofunctionality, and coclustering.*
- *Local and non-local, links, and hyperlinks.*
- *Tags, annotations, reduction, classification, coordinatization.*
- *Structuralisation and compression of redundancies.*[220]
- *Ability and tendency for repetition and imitation.*
- *Fast recognition of known, unknown, frequent, significant, improbable, nonsensical.*
- *Evaluation of degree of "playfulness" or "metaphoricity" of words, phrases, and sentences.*[221]
- *Recognition of self-referentiality.*
- *Evaluation of parameters of ability/quality of predictions:*
- *Speed, precision, specificity, rate of success, the volume of memory, and the numbers of parallel and sequential "elementary operations" employed,*

  etc.

A program that would imitate a human conversing in a natural language and that is seen as "realistic" from the ergo-perspective must be within 10^9–10^{12} bits in length. If such a program would fool somebody like Diderot, then its level of structurality must necessarily be comparable to that of the human ergo-brain and one would be justified in saying that this program *understands* what is being said.[222]

More seriously, a validity of a particular program and the resulting linguistic competence of a learner in $\mathcal{TONG}$ can be certified only by comparing the outcomes of several programs, say comprised and executed by visitors from 100 different

[218] We know that such programs are fully operational in the brains of 2–4 year old children.

[219] Relative frequencies of "events" are essential for learning a language but such concepts as "probability", "correlation", "entropy", cannot be applied to languages, without reservation.

[220] The essence of understanding is not so much extracting "useful information" but rather understanding the structure of redundancies in texts. Non redundant texts, such as tables of random numbers and telephone directories do not offer much of what is worth UNDERSTANDING.

[221] Playfulness is the first visible manifestation of "ergo" in humans (and some animal) infants.

[222] Beware of ELIZA type programs that respond to everything you say by: "You are right, it is very profound what you say. You must be very intelligent".

Universes. Conceivably, 70 among them will not be able to communicate in $\mathcal{TONG}$ with anybody, but 30 will be able to talk to each other in $\mathcal{TONG}$ and find it interesting. These 30, by definition, UNDERSTAND $\mathcal{TONG}$.

(Competent in $\mathcal{TONG}$ and linguistically naive native speakers will judge differently.)

27. Libraries, Strings, Annotations, and Colors

Libraries LIBR we have in mind may contain 10^7–10^{11} *input units* that are possibly overlapping stings of letters starting from a fragment of a word to a paragraph with a few dozen words in it.[223]

Annotation may be seen as an assignment of "*colors*" to these strings as well as connecting some units with colored edges, where these colors depict "essential properties" of the nodes or edges they are assigned to and serve as *descriptors* of *input units* in (annotated) LIBR and later on in EDI.

A color on a node u may describe a *pre-syntactic* property of a string, such as

short string, median string, long string, and *frequent string*,

but syntactic/semantic features of strings will be assigned different classes of colors from different classes.

Similarly, a color on a connective may signify a type of a geometric/temporal relation between strings, such as

overlap, contained-one-in-another, close-one-to-another, far-one-from-another, next-to-each-other, in-between, begins-with,

etc., where these "colorful concepts" come in several subcolor-flavors similarly to (yet, differently from) how it is with lengths of strings.[224]

These connectives are tied together themselves by relations between them. For instance, *closeness* between two strings often comes as *simultaneous containment* of these string into a longer string.

This is essential for enlisting and keeping in memory (pre)syntactic insertions between strings, in particular all pairs of identical words w in L.[225]

And another class of colors such as for similarity $\leftrightsquigarrow$, and for reduction arrows, may refer to a particular classifier algorithm defining/producing these arrows.

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[223]There are also larger units, call them "pages", "volumes", "shelves", but these play different roles.

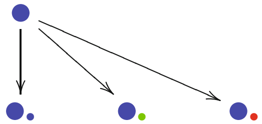
[224]The ergo-brain, that processes incoming information in parallel, divides long strings into pieces and keeps track of their mutual positions by means of binary (and ternary such as ***in-between***) relations.

Although the linear order •••••••••••••••• can be *formally* reconstructed from ***begins-with***, a baby-like learner is unaware of formal logic and cannot indefinitely iterate a single relation.

[225]The number of pairs of (short) identical strings is uncomfortably large, being *quadratic*, but identifying identical words, say on a single page, goes *linearly* in time.

This is gratifying. Squares are unacceptable except of small quantities. We happily live through a *million* seconds that make up less than 12 days of our lives. But a *trillion seconds*, that is a million squared, stretch over more than 31 000 years.

Colors themselves are structural entities but much simpler ones than incoming units such as sentences. One may think of them as simple graphs, e.g., little trees with few branches.



The library colors – about 100 in number – pass to EDI, our ergo-dictionary where they are organized into a more elaborate colored network with additional descriptor-colors, maybe up to 1000 of them.

A more difficult/interesting aspect of annotation is identifying "significant strings" (this is essentially what is called *parsing*) such as ***word**s*, ***phrase**s*, etc., and disregarding insignificant ones.[226] These also come in colors and subcolors such as ***word***$_{\text{rare}}$, for instance.

Collections, Ensembles, Sets

Collections/ensembles of linguistic units and of connections between them cannot be indiscriminately operated with, as we do with sets in mathematics for the following reasons.

1. The presence of a particular node in a network, e.g., of a particular phrase, in the long-term memory of a learner is often ambiguous.
2. Basic set-theoretic constructions, such as the union $X_1 \cup X_2$ and the Cartesian product $X_1 \times X_2$, cannot (and should not) be unrestrictedly performed in our networks.

The set theoretic language may lead you astray;[227] yet, we use fragments of this language whenever necessary.

Besides "localized" relations between strings and connectors between them, there is a large scale geometry in LIBR that is seen in the presence of (relatively large) *contextual units* such as ***page**s* and ***book**s*.

At the first stage of the annotation process these units are classified/colored by their size, where different classes must roughly fit into the corresponding frames of the *short-term, medium-term*, and *long-term memory*. Then the concept of "context" is modified and refined in the course of learning (not quite) similarly to how it happens to strings, where the true ***page**s* and ***book**s* must be either sufficiently *statistically homogeneous*, or *structurally unified*, or have *pronounced boundaries*.

Coloring Colors. It seems not difficult to make a complete combinatorial description[228] of LIBR with a few dozen (about 100?) "colors" that are *descriptors* of *basic types* of units and of connectors between them in LIBR.

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[226] An essential (but not the only) intrinsic motivation for doing this by an ergo-learner is economizing memory space.

[227] Bringing forth *random sets* and/or *fuzzy sets* may only aggravate the problem.

[228] Implementing this, i.e., achieving a proper annotation and/or parsing, is by no means easy.

But the principal issue is not so much LIBR per se but a construction of an adequate *colored network of descriptors* on the basis of a few, probably 4–8, "general rules". Eventually this network will be laid in the foundation of *EDI* – that is a *stationary model* of understanding $\mathcal{TONG}$ as represented by the library LIBR.

28. Teaching and Grading

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A universal language-learning program *PROG* is supposed to model a mind of a child and it needs only minimal help from a "teacher", such as ordering texts according to their complexity[229] and allowing *PROG* a flexible access to texts.

On the other hand, evaluation of the *quality of understanding* by *PROG* is harder (albeit much easier than designing a learning program itself), since no one has a clear idea of what *understanding* is.

Our formal approach is guided, in part, by how it goes in physics, where an unimaginably *high level of understanding* is reflected in the *predictive power* of mathematically-formulated *natural laws* that encapsulate enormously *compressed data.*

This lies in a category quite different from what we call *"knowledge"*.

[229]One may also equip *PROG* with an ability, similar to that possessed by children up to age 2–3, to resist a "bad teacher" by rejecting environmental signals that are detrimental for learning. (This ability deteriorates with age as one has to adapt to the environment in order to survive.)

For instance, ancient hunters *knew* more of how planets wander in skies than most modern people do. But understanding of this wandering depends on "compression" of this knowledge by setting it into the slender frame of mathematically formulated laws of motion.[230]

Similarly, understanding languages depends on compression of structural redundancies[231] in flows of linguistic signals, albeit this compression is not as substantial as in physics.

Besides "sheer knowledge", *understanding* should be separated from *adaptation*. For instance, an experienced rodent (or a human for this matter) competently navigates in its social environment. But only metaphorically, one may say that the rodent (or human) "*understands*" this environment.

With the above in mind, we indicate the following two mutually linked attributes of what we accept as "*understanding*".

1. *Structural compression of "information"*.

2. *Power of prediction.*

These **1** and **2** can be quantified in a variety of ways. For example, one may speak of the degree of compression versus the "percentage" of structure lost in the course of compression, while the essential characteristics of a prediction is *specificity* versus *frequency of success.*

This kind of quantification may be used for *partially ordering* "*levels of understandings*" that may suggest tests for evaluating progress achieved by a learning program *PROG* in these terms.

Another attribute of "*understanding*" that is easy to test but hard to quantify is as follows.

3. *Ability to acquire knowledge.*

For instance, a program *PROG* minimally proficient in English, would "know", upon browsing through Encyclopaedia Britannica, that cows eat grass and cats eat mice.[232]

Also, the following can be seen as a hallmark of understanding.

4. *Ability to ask questions.*

(Those whose business is UNDERSTANDING – scientists and young children – excel in asking questions.)

Besides the ability to understand, learning programs *PROG* may be graded according to their "internal characteristics", such as the *volume of memory* a

[230] Ancients astronomers came to *understand* periodicity of planetary motions and were able to make rather accurate predictions.

[231] One cannot much compress "useful information" without losing this "information", but if we can "decode" the structure of redundancy it can be encoded more efficiently.

[232] Properly responding to "*Do black cats eat fresh mice?*" instead of the plain "*Do cats eat mice?*" would need a study of a more representative corpus of English than Encyclopaedia Britannica by *PROG*.

PROG has to use, the *number of elementary operations* and *the time* needed for it to make, for instance, a particular prediction.

Eventually, full-fledged ergo-systems must contain *self-control programs*, such as an evaluation of *increase* of quality of predictions as extra information becomes available to the learner. (Such control programs are much easier to design than the core ergo-programs.)

29. Atoms of Structures: Units, Similarities, Co-functionalities, Reductions

Much of learning and understanding consists of *structuralizing* incoming flows of signals you perceive that is achieved by identifying redundancies in these flows and representing "compressed flows" of these signals in a structurally efficient way.

It is a fundamentally unresolved problem in psychology to identify *mathematical classes of structures* that would *model mental structures* built by human brains that assimilate incoming "flows of signals".

We do not know what, specifically, these structures are but a few of their ingredients are visible.

Let us make a short (and incomplete) list of four "logically (quasi)atomic constituents" of (ergo)operations applied to flows of signals with no attempt at this point to give precise definitions of these "atoms", to justify their reality, and/or to explain how one finds them in flows of signals.

1. Discretization and Formation of Units

The first step in structuralizing flows of signals is identifying/isolating *units* in these flows, where the simplest (but not at all simple) process serving this purpose is *segmentation*: Dividing a flow into non-overlapping "geometrically simple parts". (The *internal* correlations/connections in these "parts" must be significantly stronger than *mutual* correlations/connections between different "parts".)

These may be small and frequently appearing signals, such as *phonemes, words* and *short phrases* in the flow of speech or basic visual patterns such as *edges* and *T-junctions.* Also these may be as long as *sentences, internet pages, chapters in books*, or intrinsically coordinated visual images of objects such as *animals, trees, forests, buildings, and mountains.*

And even if, say, paintings and ■'s, are both regarded as "units", they are unlikely to be filed by your (visual) ergo-system in the same "units-directory".

(Processing of linguistic and visual inputs by your (ergo)brain, probably, relies on natural parsing of incoming flows of signals followed by a combinatorial organisation of the resulting "units".

The "Garden of earthly delights". Oil painting after Hierony'.
Wellcome Collection

On the other hand, the *proprioception sensory system*[233] and motor control of skeletal muscles may also depend on *continuity*, since the incoming signals may not(?) be naturally decomposable into "discrete units".)

But our ergo-structure is built from invisible *internal* units that may be grossly dissimilar to the units of incoming flows; one needs truly universal *discretizers* – "meaningful segmentation" algorithms – to discern these.

Naively, *unit* is anything that can be characterized in a few simple words, but ... these words may be of very different kinds depending not only on the intrinsic properties of such a unit, but also on how it is being processed by a particular ergo-system, e.g., a human ergo-brain.

We follow the lead of natural languages that make units of everything, *qualities, states, actions, processes,* ... by what is called *nominalization*:

everything deserving a name becomes a unit.

Example: Non-Textual Syntactic Units.[234] In order to implement this "definition", one needs to to design a program that would "understand" what "deserves" is. This is most essential for understanding languages, that unlike understanding non-linguistic arrays of signals, decisively depends on the formation of units that are *not geometric fragments of texts.* For instance, the groups of superficially dissimilar words, such as

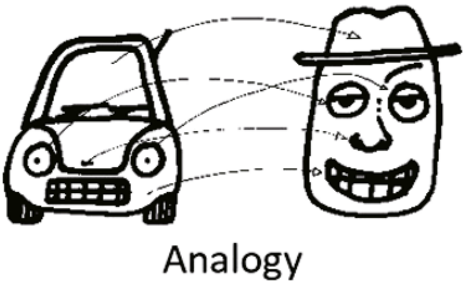
yes, no, maybe; we, us, our; big, large, huge; smelly, tasty, crunchy.

are a kind of "outlines" of such units. Identification of these is an essential aspect of passage from a library to an ergo-dictionary.

[233] This is the perception of motion, of stresses, and of positions of parts of the body.

[234] The word "syntactic" is understood in the present article as "characteristic of languages".

Analogy, Similarity, Equivalence, Equality, Sameness



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There are several *similarity relations* between units of languages/images where these relations may differ in kind and in strength.

For example, images may be similar in shape, size, color, subjects they depict, etc., and two sentences may be similar in the kind and style of words they employ, the idea they convey, or in their syntax. The strongest similarities in texts *are letter-wise equalities* of *different* strings.

There is a discrepancy between how the concept of *equality* is treated in mathematics/logic and in natural languages: We happily say

$$2 + 3 \textit{ equals } 5$$

but

$$5 \textit{ equals } 5$$

appears not very informative even to a logically indoctrinated mathematician – these two "*equal*" are not mutually equal, and the common language has no means to express this inequality. For example,

$$5 \textit{ is the same as } 5$$

does not make it look better. But this can be settled if we introduce an ergo-system in the picture, where equalities as well as weaker similarities result from certain *processes* that are qualitatively different from how one arrives at *sameness.*[235]

On Composability of Similarities. Customarily, one defines an *equivalence* as a *symmetric binary relations* on a set[236] S, denoted, say by $s_1 \sim s_2$, that satisfies the transitivity property:

$$s_1 \sim s_2 \;\&\; s_2 \sim s_3 \Rightarrow s_2 \sim s_3.$$

It is more convenient to depict equivalences (and similarities) of signals s in a category-theoretic style by arrows with "names" attached to them, such as

[235] The spirit of this is close to how different levels of "equivalence" are treated in *the n-category theory*.

[236] This definition does not cover equivalences between theories and/or between categories, because these are *are not* relations on *sets*.

$s_1 \overset{f}{\leadsto} s_2$, where one think of such an arrow as an "*implementation of* $\sim$" by some "logical/computational process", e.g., by some co-clustering algorithm.

Then one may compose arrows

$$s_1 \overset{f}{\leadsto} s_2 \overset{g}{\leadsto} s_3 \text{ with the composition denoted } s_1 \overset{f\square g}{\leadsto} s_3.$$

This allows one, for instance, to say that

the composition $f \square g$ of two "strong similarities" f and g is itself a "weak similarity".

Also one can now speak of certain equivalences f and g, e.g., one in color and another one in size, being *incomposable*.

3. Classification, Reduction, Clustering, Compression

Equivalence relations E on a set S go hand in hand with partitions of this S into the corresponding *equivalence classes* that can be conveniently described via the *reduction map* $R = R_E$ from S *onto* a C, such that

$$s_1 \underset{E}{\sim} s_2 \textit{ if and only if } R(s_1) = R(s_2).$$

However, implementations of a *binary* relation $s_1 \underset{E}{\sim} s_2$ and of a *unary* operation $R(s)$ are quite different from a working ergo-system point of view.[237]

It is much harder to record $\approx N^2/2$ bits encoding an equivalence relation on a set S with N units, than $\approx N \log N$ bits needed for defining $R(s)$; *similarities* and *reductions* must be treated separately.

An essential feature of *reductions* from our perspective is *compression of information* and

creation of new units c from the original units s, that are $c = R(s)$.

A more general and less cleanly-defined class of operations is called *clustering*, that is based on similarities that are not sharply defined and are not perfectly *transitive*, unlike what is usually required of "equivalence".

The tautological map $R : s \mapsto c$ associated to a given clustering that assigns to each member s of S the cluster c in S that contains s (this R may not be defined for all s) is still called the *quotient map* or *reduction* from the original set S to the set C of clusters. The reduction that defines co-clustering is an instance of this.

Compression, Morphisms, Functors. Besides the above, there are reductions of quite different type that correspond to "non-local" *compression with a limited loss of information*, where one forgets non-essential information in a text, or in a visual image, while preserving the significant structure/content of it; this is a hallmark of understanding.

[237] This is discussed at length in the context of cognitive linguistics by George Lakoff in "*Fire Women and Dangerous Things*" where *classification* is called *categorization*.

It may happen, of course, that a text has little redundancy in it, such as a telephone directory, for instance. Then no significant reduction and no understanding of such text is possible.

In fact, "perfect texts" with no redundancy in them are indistinguishable from random sequences of symbols, but every meaningful text T admits many reductions, depicted by arrows, say $T \overset{r'}{\to} T'$, $T \overset{r''}{\to} T''$, where the bulk of the process of understanding a text consists of a multi-branched random cascade of such reductions.

An example of a significant commonly used reduction is making a *resume* or *summary* of a text. Also giving a *title* is an instance of a reduction – a *terminal reduction*: You cannot reduce it any further without fully degrading its structure.

If we agree/assume/observe that consecutive performance of reductions, say $T_1 \overset{r_{12}}{\to} T_2$ and $T_2 \overset{r_{23}}{\to} T_3$, make a reduction again, denoted $T_1 \overset{r_{13}}{\to} T_3$, also written as *composition*

$$r_{13} = r_{12} \circ r_{23},$$

then reductions between texts can be regarded as *morphisms*, of the **category** (in the mathematical sense) of *texts and reductions* where, strictly speaking the word "*reduction*" suggests these arrows r being *epimorphisms*, i.e., they add *no new information* to texts they act upon.

It may be amusing to encode much (all?) information about a language $\mathcal{L}$ – *syntax*, *semantics*, *pragmatics* – in terms of such a category $\mathcal{R} = \mathcal{R}(\mathcal{L})$ of reductions in $\mathcal{L}$, with translations from one language to another, $\mathcal{L}_1 \rightsquigarrow \mathcal{L}_2$, being seen as *functors* between these categories; but it is dangerous to force categories into languages prematurely.

Reduction and Agglomeration of Similarities. There are circular relationships between similarities of different types and/or of different strengths. For instance two signals s_1 and s_2 that have equivalent or just strongly similar reductions may be regarded as weakly similar.

Conversely, if there are "many independent" weak similarity relations between s_1 and s_2 then s_1 and s_2 are strongly similar and possibly equal.

To see what we mean, imagine you have two books, each approximately 200 pages long. Choose any way you want the numbers of pages, say 150 altogether, and count how many times "*the*" appears on each chosen page i of the books 1 and 2.

A similarity of these two "*the*" *contents* N_1 and N_2, such as

$$N_1 - 2 < N_2 < N_1 + 2,$$

for a single pair of pages is not informative, but if this relation holds for all 150 pairs of your chosen pages, you bet that 1 and 2 are copies of the same book.

4. Co-functionality

Some units in a text T or in another kind of flow of signals form relatively tightly knit groups where we say that these units *perform a common function*.

A priori, co-functionality is not a binary relation (albeit it is helpful to assume so when defining *co-clustering*); it can be, however, made binary by giving "names" to these "*functions*" and by regarding functions as new kinds of units.

Then we say that *unit s performs function f* and depict this by a directed edge $s \longleftarrow f$. Alternatively, we depict f-co-functional units as being joined by f-colored edges $s_1 \underset{f}{\longleftrightarrow} s_2$.

Connections between Units: their Identification, Nominalisation and Classification

Deciding which units are essentially independent and which have non-trivial *connections/relations* between them is one of the first priorities of an ergo-system that must be accomplished by several algorithms, that can be called *connectors*.

Such connectors, being themselves particular kinds of units, need to be classified by universal algorithms as befits all decent units, where the coarsest classification would separate the following three classes of relations.

- *Similarity*.
- *Cofunctionality*.

(The latter is common not only for pairs but also for triples and possibly quadruples of units that *perform together* certain functions. This "togetherness" is manifested by systematic co-appearance of the corresponding units.)

- *Reductions*.

Warning. Treating relations on an equal footing with initial units opens a pandora's box of self-referentiality in our ergo-system(s). Apparently, this is necessary for the kind of ergo-behaviour we want to achieve but this also inoculates our systems with logical paradoxes. Somehow one has to strike a balance between dumbness and madness of ergo-programs.

After we have developed algorithms for the structural analysis of "incoming flows of signals" along the above guidelines, we shall be able(?) to decide if there is *some unknown* "*else*" within the human mind crucially involved in the "learning to understand" process that is fundamentally different from formation of units, their classification, and their combinatorial organisation according to their connections and interactions.

The fundamental difficulty we face here appears when we attempt to structuralize not only incoming flows of signals, but also those *created and circulating within* learning system itself, where these "internal flows" are not, at least not apparently, grounded on any structure similar to what underlies "true flows": The linear (temporal or spatial) order between signals.

The data obtained in this regard by neurophysiologists and psychologists do not tell us, at least not directly, how to proceed – we take our cues from what mathematics has to offer.

But while thinking mathematically, we also need to keep in mind possibilities and limitations of the brain, ergo-algorithms must be "broad and shallow": They cannot have many (say, more than 5) *consecutive operations* on each round (unit) of computation (that, roughly, corresponds to what we routinely do on a one-second time scale); yet, allowing several hundred (thousand?) operations running in parallel.[238]

30. Fragmentation, Segmentation, and Formation of Units

Textual units, such as particular fragments of incoming signals,[239] e.g., particular strings of letters such as "*words*",[240] or some distinguished regions in visual fields, such as "*perceived objects*" or "things",[241] lie at the heart of all *combinatorial models of understanding*. Albeit one can hardly give a comprehensive definition of such units, or of *signal-units* in general, these are frequently recognizable by the following property

> *The probability of encountering a textual unit u among a multitude of other signals in the same category as u* (here "category" means *class*) is *significantly greater than the product of probabilities of* "*disjoint parts of u*".

For instance the word "probability", which has 11 letters in it, may, a priori, appear only once or twice in a library with a billion books ($<< 26^{11}$ letters) in it.[242]

This does not work quite so nicely for short words: Scrabble dictionaries offer ≈1000 three-letter English words and ≈4000 four-letter words where many of them, e.g., *qat* (an African plant) or (to) *scry* (to practice crystal gazing) appear rarely, but the improbable frequency of such a word may be seen in the appearance of several copies of it in a single volume, or even on the same page.

[238] This parallelism is the "technical reason" why our basic mental (ergo)processes are inaccessible to our sequentially structured conscious minds.

[239] We temporarily ignore overlaps between fragments such as "*hard to see*" and "*to see it*" in the unit-phrase "*hard to see it*". ("*Hard to*" makes a perfect "unitary uttering"; yet this is a weaker unit than "*hard to see*".)

[240] A textual unit may be "disconnected", e.g., it may consist of two (more?) strings separated by other strings in a text. This happens, for instance, to separable prefixes in German that are moved to the ends of sentences. Also this **is** *not exceptionally rarely* **seen** in English.

[241] The rigid concept of *object-unit* modifies by classification/reduction and applies to "things" that come in many shapes such as words with flexible morphological forms, the human body, or to something inherently random such as an image of a tree with multiple small branches. When our eye looks at such a tree, our mind, conjecturally, sees (something like) a *branch/shape distribution law* rather than the sample of such a distribution implemented by an individual tree.

[242] The number of different books in the world is estimated at about 100 million.

The abnormal frequency alone, however, does not define units: The string "obabili" appears at least as often as the full "probability"; thus, one has to augment the "definition" of a unit by the following

completeness/maximality condition: If a string s is a unit, then larger strings $s' \supsetneqq s$ are *significantly* less probable than s.

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Segments and Boundaries. Fragmenting texts into units is naturally *coupled* with the process of *segmentation* that is introduction of *division points* that make boundaries of string-units in texts.[243]

The determination when the position d in a string S between two letters may be taken for a division point depends on the strings s "to the left" and "to the right" from d in S, where such a string, say s_{left}, being a unit is an essential indication for d being a division point.

But it may also happen that there are no such clear cut units next to d in S but there is a 20-letter string S' somewhere else in the library that contains isomorphic copies of five-letter strings to the left and to the right from d and such that the corresponding d' is recognizable as a division point in S'. Then we may accept d as a division point in S and to use this for identifying previously unseen units in S.

The coupled *fragmentation + segmentation* is a multistage process, each step of which is a part of the learning transformation $PROG$ on a certain space of pairs $(Frag, Seg)$ that incorporates into the full "understanding space" on the later stages of learning.

This process must comply with the "*please, no numbers*" *principle*: The program $PROG$ we want to implement must function similarly to an infant's brain that, unlike an extraterrestrial scientist, has very limited ability of counting and of manipulating large numbers (e.g., frequencies) as well as small ones (e.g., probabilities).

[243]Boundaries of so-called "words" are marked in most written languages by white spaces, but phrases and sentences are pinched between division punctuation signs. But we pretend to be oblivious to this for the moment.

This is achieved by consecutive "internal fragmentation" of the process *PROG* itself into a network of simple processors/directories where, they all, individually, perform (almost identical) "baby operations" with the global result emerging via communication between these processors.

Classification of Words and Partitions into Sentences. Segmentation of texts into strings with more than 2–3 words in them is impossible without preliminary syntactic classification of *basic units* – words and short phrases.[244] But when such a classification is performed the number n of basic units u – this n is about 10^5–10^6 in English – is reduced to much smaller number $\underline{n}$ of classes $\underline{u}$, realistically with $10 \leq \underline{n} \leq 30$. Then a library with N basic units in it would allow one to reconstruct the rule of formation of strings of length about $\log_{\underline{n}} N$. For instance, if we classify with $\underline{n} = 20$, then a modest library with 10^9–10^{10} basic units in it[245] gives an access to 6–8 basic unit long strings, for $\log_{20} 1.3 \cdot 10^9 \approx 7$ that may allow an automatic discrimination between *admissible* and *nonsensical* strings up to, maybe, 12 words in length. Then *generation* of *meaningful strings* becomes a purely mathematical problem.

Gross Contextual Segmentation. In the spoken language, utterings are divided according to when, where, and who is speaking to whom, but texts in written languages are organized into paragraphs, pages, books, topics, libraries, with a similar arrangement of pages on the web.

These partition structures are essential for making a statistical analysis of languages; conversely, texts can be classified/partitioned according to relative frequencies of short-range structural patterns, e.g., basic units, present in them.

31. Presyntactic Morphisms, Syntactic Categories, and Branched Entropy

Deep linguistic structures display some *approximate category-theoretic* features, e.g., *abridgements* may be seen as *semantic epimorphisms*, or as *functors* of a kind rather than mere "morphisms".

Then translations from one language to another come as functors between categories (2-*categories* if abridgements are regarded as functors) of languages, where the category-theoretic formalism should be relaxed to accommodate imprecision and ambiguity of linguistic transformations.

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[244] The single word in *Bininj Gun-wok* (an aboriginal language in northern Australia)

abanyawoihwarrgahmarneganjginjeng

corresponds to *I cooked the wrong meat for them again*, [13].

[245] There are about 100 basic units on a page, 10^4–10^5 of such units make a book, a 10 000 book library comprises $\approx 3 \cdot 10^9$ units, and the world wide web may contain up to 10^{12} basic units of the English language.

But we shall be concerned at this point with the following more apparent combinatorial category-like structure that is universally seen in all kinds of "flows of signals".

Let a *library* L in, say English, language $\mathcal{L}$ be represented by a collection of tapes with strings s of symbols, e.g., letters or words, written on them, where many different tapes may carry "identical" or better to say *isomorphic* strings with the notation $s_1 \simeq s_2$, with the equality notation $s_1 = s_2$ reserved for the same string in the same location on the same tape.

Let arrows $s_1 \hookrightarrow s_2$ correspond to *presyntactic insertions* between strings, i.e., where such an arrow associates a substring $s_1' \subset s_2$ to s_1, where $s_1' \simeq s_1$.

We assume our strings are relatively short, no more than 10–20 words of length: This is sufficient for describing any "library" since every 10-word-long string *uniquely* (with negligibly rare exceptions) extends (if at all) to longer strings, since the total number of strings in any language is well below $100^{10} << n^{10}$ for n being the number of symbol-words in a language.[246] As for L one might think of something with the number N of words in it in the range 10^6–10^{12}.

The resulting ***category*** $\mathcal{C}_{\hookrightarrow} = \mathcal{C}_{\hookrightarrow}(L)$ *carries full information about* L.

Justification

[+] *Invariance.* $\mathcal{C}_{\hookrightarrow}$ is invariant under changes of "alphabets" – names of the symbols.

[++] *Universality and Robustness* The categorical description of languages satisfies the most essential ergo-requirement that is UNIVERSALITY.

For instance, *spoken languages* can be similarly described in categorical terms, where, unlike written languages, the arrows must correspond to *approximate* insertion relations between auditory or visual patterns.

In fact, allowing *approximate* presyntactic insertions with *sequence alignments* (with a margin of error 5–10%) in place of syntactic isomorphisms between strings would enhance the *robustness* of categorical descriptions of *written* languages as well.

Features

[*] *Non-locality.* The $\mathcal{C}_{\hookrightarrow}$-description of libraries depends on comparison between strings that may be positioned mutually far away from each other in texts.

[**] *Long-Term Memory.* This comparison between strings, depends on the presence of a structurally organized, albeit in a simple way, *memory* within the learning program.[247]

[246] Never mind the saying: "*there are infinitely many possible sentences in a natural language*".

[247] Conceivably, this organisation corresponds to how languages are perceived by their principal learners (1–4 year old children), where the $\mathcal{C}_{\hookrightarrow}$-categorical organisation of memory is the "ground level" of what we call "understanding" of $\mathcal{L}$.

Redundancy and excessive local complexity of $\mathcal{C}_{\hookrightarrow}$

[−] The full category $\mathcal{C}_{\hookrightarrow}(L)$ contains many "insignificant" arrows, e.g., insertions of *single letters* into ten-word sentences and arrows between "non-linguistic" strings, such as "tic stri".

This can be corrected by

*allowing only **textual units** for objects in* $\mathcal{C}_{\hookrightarrow}$

and by

selecting a ***representative subdiagram*** $\mathcal{D}_{\hookrightarrow} \subset \mathcal{C}_{\hookrightarrow}$.

Such a diagram $\mathcal{D}_{\hookrightarrow}$ (that is a *network* of directed *arrow-edges* between *strings for vertices*) *must generate* (most of?) $\mathcal{C}_{\hookrightarrow}$ as a monoid, and also it must be "small", e.g., being a *minimal* subdiagram generating $\mathcal{C}_{\hookrightarrow}$.

(There is no apparent natural or canonical choice of $\mathcal{D}_{\hookrightarrow} \subset \mathcal{C}_{\hookrightarrow}$, but it may depend on the order in which the learner encounters texts in the library.)

[−+] *Pruning and Structuralizing* $\mathcal{D}_{\hookrightarrow}$. No matter how you choose $\mathcal{D}_{\hookrightarrow}$ it has *too many arrows* issuing from certain (relatively short) strings s, where the number of such arrows grows with the size of a library. Thus, in order to comply with the principles of ergo-logic, our learning algorithms must automatically reorganize $\mathcal{D}_{\hookrightarrow}$ in order to correct for this excessive branching. This is achieved by operations of *reduction*[248] applied to (the sets of) strings and arrows.

Categories $\mathcal{C}^{\hookrightarrow\downarrow}$ and Diagrams $\mathcal{D}^{\hookrightarrow\downarrow}$ of Annotated Texts

If the texts in a library are annotated with tag-strings s' that are written on several levels over original strings s, then the category with "horizontal" arrows $s'_1 \to s'_2$ is augmented by the "vertical" *position arrows* $s'' \downarrow s'$ saying that s'' lies over s', where such "mixed categories" and their representative subdiagrams are denoted by $\mathcal{C}^{\hookrightarrow\downarrow}$ and $\mathcal{D}^{\hookrightarrow\downarrow}$.

The presence of vertical arrows serves two purposes.

[1] Vertical arrows significantly *increase the connectivity* of diagrams since *a bound on the number* of tag-words on the *high levels* of annotations yields the existence of *many* horizontal (syntactic insertion) arrows between strings on these levels that were not present on the lower levels.

[2] And

the notion of a representative diagram $\mathcal{D}^{\hookrightarrow\downarrow}$
is modified in the presence of vertical arrows

by replacing many horizontal arrows issuing from lower-level strings in an annotated text by the corresponding arrows on the higher levels where the "low-level information" is encoded by (inverted) vertical arrows. Thus one (partly) compensates for the excessive branching of $\mathcal{D}_{\hookrightarrow}$.

[248]This is also called *clusterisation, classification, categorisation, factorisation.*

Arrow of Time. The category $\mathcal{C}_{\hookrightarrow}$ is unaware of the time direction in linguistic strings, but this direction in (written or sound recorded) strings can be, probably, reconstructed by the predominantly *backward orientations* of self-references in texts.

Structures in Symbols. In our categorical description (the alphabet of) the basic symbols, say letters, carry no internal structure of their own. But in reality letters in alphabets are non-trivially structured in agreement with one of the ergo-logic principles that allows no unstructured set of objects with more than three–four members in it. I am not certain what one should do about it.

Dimension of Vision. Visual signals[249] are customarily recorded on *two-dimensional* backgrounds such as on photographs and/or eye retinas, where the extra dimensions of depth and of time (in moving pictures) carry only auxiliary information. The morphisms $s_1 \hookrightarrow s_2$ here correspond to similarities between visual patterns s_1 and subpatterns in s_2.

But, probably, a significant part of visual perception is *one-dimensional* being implemented/encoded by the neurobiology of *saccadic eye movements*. This suggests unified algorithms for learning to see and for learning to speak.

Syntactic from Presyntactic

Eventually, we isolate strings (sometimes pairs of strings) that serve as *textual units* and also we identify *significant insertions* between them that we call *syntactic insertions.*

Linguistic 2-Spaces $\mathcal{P}_{\hookrightarrow} = \mathcal{P}_{\hookrightarrow}(L)$ and $\mathcal{P}^{\hookrightarrow\downarrow}$

Let us represent strings from a given library L by line segments of lengths equal to the numbers of letters in them. Attach rectangular 2-cells to the disjoint union of all these strings, where these "rectangles" are Cartesian products $s \times [0,1]$, with s being some strings/segments of length ≥ 5 letters each and where the attachment maps are syntactic insertions from the segments $s \times 0$ and $s \times 1$ to some string segments S_0 and S_1, such that the images are *maximal mutually isomorphic* (i.e., composed of the same letters) substrings in S_0 and S_1.[250]

In fact, it is more instructive to use the maps corresponding *not to all* syntactic insertions in the category $\mathcal{C}_{\hookrightarrow} = \mathcal{C}_{\hookrightarrow}(L)$ but only to those from a *minimal* diagram $\mathcal{D}_{\hookrightarrow} \subset \mathcal{C}_{\hookrightarrow}$, that *generates* all morphisms from $\mathcal{C}_{\hookrightarrow}$ on strings of length≥ 5.

Then the resulting two-dimensional cubical (rectangular) polyhedron $\mathcal{P}_{\hookrightarrow} = \mathcal{P}_{\hookrightarrow}(L)$ adequately encodes the library L and if L is sufficiently large, this $\mathcal{P}_{\hookrightarrow}$

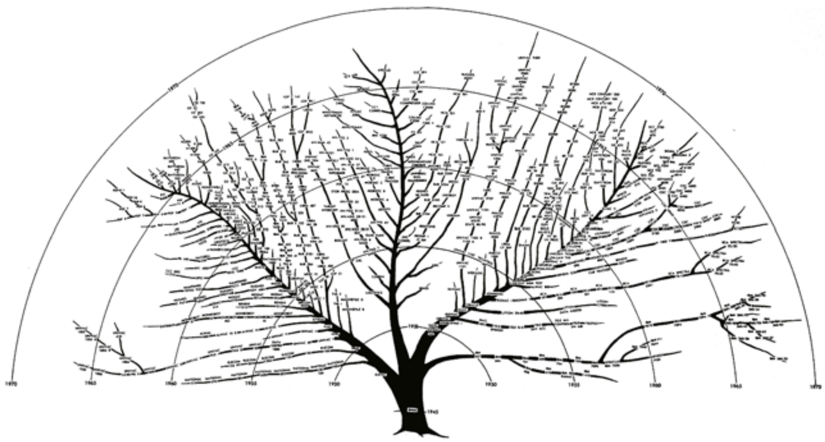
[249] There is a demarcation line separating visual structures of *Life* (plants, animals, humans, human artefacts) from those of *non-Life* (stretches of water, rocks, mountains). These two classes of images are, possibly, treated differently by the visual system.

[250] Our ad hoc bound *length* ≥ 5 serves to eliminate/minimise the role of "meaninglessly isomorphic" substrings, (e.g., of individual letters) where the same purpose may be implemented by a natural constraint on strings and gluing maps.

carries all structure knowledge of the corresponding language $\mathcal{L}$ with segmentation into basic units – words and short phrases made visible.

If one deals with the category $\mathcal{C}^{\hookrightarrow\downarrow}$ corresponding to an *annotated* library, or with a subdiagram $\mathcal{D}^{\hookrightarrow\downarrow} \subset \mathcal{C}^{\hookrightarrow\downarrow}$, then one attaches "vertical" rectangles along with "horizontal" ones, where the horizontal rectangles are associated to the arrows $s'_1 \hookrightarrow s'_2$ and the vertical ones to the arrows $s'' \downarrow s'$.

Branching Entropy



© John Redant & DigiBarn Computer Museum,
http://www.digibarn.com/stories/desktop-history/compyear-bushytree1970m.gif

Extensions of a *string-unit*, s, e.g., of a word, by short units t following next after s in a library L define a *probability measure* on these t for

$$p_{\vec{s}}(t) = p_{L,\vec{s}}(t) = N_L(st)/N_L(s),$$

where $N_L(s)$ and $N_L(st)$ denote the numbers of occurrences of the strings s and of st respectively in L.

The collection of numbers $\{p_s(t)\}$ indexed by t serves as an indicator of variability of usage of s in the texts in the library L, where it seems reasonable to use not all t but only a collection T of unit-strings (words) t corresponding to the roughly 10 (it may be something between 3 and 50, I guess, that need be determined experimentally) *largest numbers* among $p_{\vec{s}}(t)$.

The standard invariant of the probability space $\{p_{\vec{s}}(t)\}$ that reflects variability of p and regarded as an invariant of s is the (*one step forward*) *entropy*

$$\overrightarrow{\mathrm{ent}}(s;L) = -\sum_{t\in T} p_{\vec{s}}(t)\log p_{\vec{s}}(t).$$

Similarly, one defines $\overleftarrow{\mathrm{ent}}(s;L)$ via left extensions ts of s as well as the corresponding invariants reflecting relative frequencies of "double extensions" of s that are t_1t_2s, t_1st_2 and st_1t_2.

Such entropies apparently are quite different for the strings "*birds-fly*" and "*pigs-fly*"[251], but "*bats-fly*" will be close to "*birds-fly*" in this respect. (This is more pronounced for extensions not of the strings themselves but of their "syntactic variations".)

32. Similarities and Classifications, Trees and Coordinatizations

Many (most?) linguistic "units", are classes of other units. For instance, *words* are, in reality, *equivalence classes of strings containing these* words, rather than mere "spell-strings". For instance, the two collections of strings

[**bats-eat**]: *bat-with-flapping-* ..., *bats-from-* ..., *bats-are-present-* ..., *vampire-bat*, *bats-catch-* ..., *inoculation-of-bats*, *bat-captured* ...

[**bat-hits**]: *training-bat*, *used-bats-on-sale*, *made their own bats*, *increase-your-bat-* ..., *throws-his-bat*, ... *-bats-per-game*, *raised-a-bat* ...

represent two different "bat" class-words.[252]

Classifications are often (but not always) achieved by means of *similarity* and/or *equivalence* relations R that, besides *similarity* and *equivalence*, reflect the ideas of

"sameness", "identity", "equality", "isomorphism", "analogy", "closeness", "resemblance",

where such relations R are regarded as *higher-order units* and are themselves subjects to further classification. For instance, the similarity $\bullet \sim_1 \circ$ is different from $\bullet \sim_2 \blacksquare$ as well as from $\triangle \sim_3 \square$, where $\sim_2$ and $\sim_3$ are similar themselves and dissimilar from $\sim_1$.

Not all similarities lead to what may be called "true classification", partly because the "equivalence axiom" $A \sim A$ is not satisfied in ergo-logic. Indeed you would become mad if you fill your brain with $A \sim A$ for all A in your head.

Also some similarity concepts are applicable only to small groups of objects, such as what brings together

$$\{\textit{sweet, bitter, salty, sour, tangy}\},$$

and that do not extend to a majority of words.

[251] The two strings have comparable frequencies on Google.

[252] Non-existence of the string "*bats eat and hit*" shows how far apart the two classes are but ambiguous strings such as "*hit by a flying bat*" effectuate "quasi-poetic bridges" between the two classes.

On the other hand, there are more – about a dozen – different class-words spelled "bat", that are, essentially, subclasses of [**bat-hits**].

Another kind of groups of words having much in common that may or may not be regarded as true classes are those of *morphological word forms* such as

$$\{\textit{works, worked, working}\} \quad \text{or} \quad \{\textit{white, whiteness, whiten}\}.$$

On the other hand, traditional parts of speech (*verb, noun, adjective*, … , etc.) represent typical *classes of words*; also division of words into "common" and "rare" is essential despite being ambiguous.

The two common classification structures are as follows.

1: Classification as a Tree. This may be seen as a sequence of partitions of word units into finer (smaller) classes, where the rule defining each following partition depends on the previous (coarser) one.

A linguistically rather artificial instance of this, is classification/positioning of words in alphabetically organized dictionaries.

A more significant example is where the first partition divides words into the two classes **A** and **B**:

A. Class of content words: {*nouns*, (most) *verbs*, *adjectives*, *adverbs*},

B. Class of function words: {*articles, pronouns, prepositions, etc.*}.

The classes of the second partition are obtained by subdividing words into "parts of speech".

Then the third partition may divide content words according to their "overall meanings", e.g., nouns according to whether they represent "physical objects" or "abstract concepts", etc.

2: Classifications by Coordinates.[253] These are given by several *coordinates* that are functions on objects we try to classify, where definition/determination (but not the value!) of each coordinate does not depend on the rest of the other coordinates.

The classes are formed by assigning particular values to some coordinates.

For instance, one may have the following functions $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, $\mathbf{c}_4$ on phrase-units u.

- $\mathbf{c}_1(u)$ takes values *long*, *medium*, *short*, depending on whether u has at most 4, between 5 and 8, or more than 8 word-units in it.
- $\mathbf{c}_2(u)$ takes values *yes* or *no* depending if u contains a *content verb* in it.
- $\mathbf{c}_3(u)$ assigns the *key word* w in u to u.
- $\mathbf{c}_4(u)$ is the expected age group (3–6, 7–11, 12–…) of a child who is able to understand the phrase u.

In general, coordinatization "imbeds" a potentially large set U of units to the *coordinate space* that is the Cartesian product of several small sets.

Our goal is formulating *universal* classification rules *a priori, applicable to all kinds of strings*, as well as differently-structured signals that would be as good, eventually better, than classifications based on "meaning".

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[253] The kind of coordinates we present below are called *features* in the machine learning community.

33. Clustering, Biclustering, and Coclustering

There can be no isolated sign. Moreover, signs require at least two Quasi-minds.[254]

Charles Sanders Peirce

Suppose certain pairs of points in a set V of "units" are connected by edges, graphically $\circ\!\!-\!\!\!-\!\!\bullet$, that represents a certain *resemblance* between these points regarded as vertices of a graph on the vertex set V.

Simple clustering of such a V is a partition of V into subsets, called *clusters*, such that (the numbers of) the connections between members of clusters are stronger (more numerous) than interconnection between different clusters.

The archetypical clustering is division of a graph into its *connected components*, but, in general, there is no mathematical clustering recipe applicable to all graphs – after all, many graphs are non-clusterable.

On the other hand, many "resemblance graphs" in life admit more or less non-ambiguous dissection into clusters.

Simple clustering often applies to sets V with a distance (or distance like) function $d(v_1, v_2)$, e.g., to a subset in the n-space with the *ordinary Euclidean* metric.[255]

Biclustering is more interesting than simple clustering. For instance, imagine a language that admits a simple general definition of *word-unit* and where there is a universal rule for identification of *word boundaries*. (In real life defining what is a word and devising an algorithm for identifying them in a flow of signals is by no means easy.)

Biclustering is a classification of words according to their *functions*: Two words w_1 and w_2 are regarded as *functionally similar* if the other words with which they systematically "cooperate" themselves tend to be similar.

[254]Mystifying but inspiring.
[255]This allows one to blissfully use precooked formulas from the book.

The condition

w_1 *is similar to* w_2 *if coworkers of* w_1 *are often similar to coworkers of* w_2

may strike you as being circular. But then you rewrite it as

w_1 *is similar to* w_2 *if coworkers of* w_1 *are often similar to coworkers of* w_2.

Now, it points toward an iterative process (algorithm) that transforms a preliminary clusterization to a more advanced one.

It is often difficult to define and/or to identify *togetherness* of "doing something" for pairs (or larger groups) of words, but it is relatively easy to decide, without any reference to "meaning" or "function" whether two given words, say w_1 and w_2, *often* come *close together*, or, on the contrary they come close relatively rarely.[256]

Both, "*close together*" and "*often*" are variable, where the latter must be adjusted to the former: What is regarded *often* for coming next to each other will be considered rare for the simultaneous presence of these words on the same page in a text.

Granted a specification of "*close*" and "*often*" one arrives at what is called the *co-occurrence graph* G on the set W of words,[257] where w_1 is joined with w_2 by an edge if the two "*often come close together*".[258]

The remarkable fact is that such graphs, if they come from "real life", have huge redundancy in them – they are very far from anything "random".

More specifically, such graphs G typically admit *approximate reductions* to certain much smaller graphs $\underline{G}$.

On terminology

Classification is our preferred word for what is called *categorization* in linguistics and in psychology: The division of "objects" into classes.

A classification achieved by some reduction $G \to \underline{G}$ is called *co-clustering* in linguistics and *bi-clustering* as well as *two-mode clustering* in data mining and in bioinformatics, where one says *clusters* rather than of "classes".[259]

We use the term "coclustering" where functional cooperation may involve more than two units and reserve "biclustering" for the above case of binary cooperation.

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[256] This pre-assumes that we know what it means to be "*same*" for words positioned at *different* locations in flows of speech or in written texts.

[257] We assume here that words constitute *sets*.

[258] Since we are now firmly on a mathematical ground, ambiguity of "*often come close together*", or vagueness of terminology on a preliminary stage of exposition, in general, poses no danger. Firstly, because it is *not disguised* as being something precise, and secondly because mathematics promotes crystallization of valid ideas into logically solid ones.

This is different from what happens to almost all speculative considerations fulled by "raw intuition" unaided by mathematics.

[259] This kind of analysis, probably, has been used in other branches of science/statistics under different names that makes it hard to find out when and by whom this idea was originally introduced. Not impossibly, this was understood and implicitly used by Aristotle.

Biclustering applies not so much to graphs but rather to *functions* in two variables, $G(u,v)$, where the domains U of u and V of v do not have to be equal. (Graphs are represented by 2-valued *edge/no-edge*, or $\{0,1\}$ for brevity's sake, functions.) Namely,

⋆ *reduction of* $G(u,v)$ to a function $\underline{G} = \underline{G}(\underline{u},\underline{v})$ that is defined on a pair of smaller, often significantly smaller, sets $\underline{U}$ and $\underline{V}$, is a pair of maps from U onto $\underline{U}$ and from V onto $\underline{V}$, say $P^{\downharpoonleft} : U \to \underline{U}$ and $Q^{\downharpoonleft} : V \to \underline{V}$, written as

$$u, v \rightrightarrows \underline{u} = P^{\downharpoonleft}(u), \underline{v} = Q^{\downharpoonleft}(v),$$

such that the composition, sometimes called *superposition*, $G^{\downharpoonleft}(u,v)$, of the functions $\underline{G}$ and $R^{\downharpoonleft}$, that is

$$G^{\downharpoonleft}(u,v) = \underline{G}(\underline{u},\underline{v}) = \underline{G} \circ P^{\downharpoonleft} \& Q^{\downharpoonleft}(u,v) = \underline{G}(P^{\downharpoonleft}(u), Q^{\downharpoonleft}(v))$$

provides a "*good approximation*" to the function $G(u,v)$.

(If $U = V$ and $G(u,v) = G(v,u)$, then one may take $\underline{U} = \underline{V}$ and $P^{\downharpoonleft} = Q^{\downharpoonleft}$. But if $G(u,v) \neq G(v,u)$ then $\underline{U}$ and $P^{\downharpoonleft}$ are not necessarily equal to $\underline{V}$ with $Q^{\downharpoonleft}$ even if $U = V$.)

⋆ *Clusters* (classes) are the subsets in U and in V corresponding to $\underline{u}$ and $\underline{v}$ via $u, v \rightrightarrows \underline{u}, \underline{v}$, i.e., the subsets of those u in U and v in V for which $P^{\downharpoonleft}(u) = \underline{u}$ and $Q^{\downharpoonleft}(v) = \underline{v}$.

"*Approximation*" here is different from the above positional closeness of words. It depends on what kind of function G is and where its values lie. In our examples, besides being a two-valued function it may be *three-valued*, saying whether u and v cooperate *strongly, weakly*, or *not at all*. Also it may be a *number-valued* function with $G(u,v)$ being the relative frequency of co-occurrence of u and v.

In either case, the range of the function, denote this range by I, must be equipped with a *metric* measuring the distances between different values.

If I equals a set of positive numbers, then one takes the absolute value $|i_1 - i_2|$ for such a distance, and if I is an "abstract" two or three point set, one may implement these by numbers, say by $\{0,1\}$ and $\{0,1,2\}$, and use the distance $|i_1 - i_2|$ again.

And then *closeness* may be defined relative to the *Hamming distance* (also called l_1-*distance*)

$$\mathrm{dis}t(G, G^{\downharpoonleft}) = \sum_{u \in U, v \in V} |G(u,v) - G^{\downharpoonleft}(u,v))|$$

To see how this works, let

- $U = V$ be a set that is comprised of 100 000 words.
- u and v in W be regarded as "*cofunctional*" if v goes *right after* u.
- "*often*" means "*at least ten times*".

The function G corresponding to this can be seen as a (non-symmetric) $100\,000 \times 100\,000$ matrix with $\{$*often*, *rare*$\}$ entries. A reliable evaluation of the

values $G(u,v)$, that are the entries of this matrix, needs a library of more than $10^{11} = 10 \cdot 10^{10}$ words in it.[260]

But it may happen, and it does often (albeit approximately) happen in "real life", that this huge matrix is (approximately) determined by something much smaller, say by a 300×300 matrix, where you need only $90\,000 < 10^5$ entries to fill in a description of which needs only

$$90\,000 + (2 \log_2 300) \cdot 10^5 < 2 \cdot 10^6 \text{ bits}$$

instead of the original 10^{10} bits.

And biclusterisation serves for achieving such simplification by reduction of your big matrix/function G to an $\underline{G}(\underline{u},\underline{v})$ that is defined on a set $\underline{U} \times \underline{V}$ with $300 \times 300 = 90\,000$ elements in it.

Quasi-Uniqueness of $\underline{G}$ and $P^{\downharpoonleft}\&Q^{\downharpoonleft}$

The existence of a reduction $G \rightsquigarrow \underline{G}$ is, a priori, *extremely unlikely* even if we do not require $G^{\downharpoonleft}(u,v) = \underline{G} \circ P^{\downharpoonleft}\&Q^{\downharpoonleft}(u,v)$ to be an especially fine approximation of $G(u,v)$.

Therefore, even if a certain $G^{\downharpoonleft}(u,v)$ delivers only a rough approximation to $G(u,v)$, the corresponding $\underline{G}(\underline{u},\underline{v})$, $P^{\downharpoonleft} : U \to \underline{U}$ and $Q^{\downharpoonleft} : V \to \underline{V}$ will be (essentially) *unique with an overwhelming probability.*

But if the sets $\underline{U}$ and $\underline{V}$ are small, say consisting of 2–4 elements, then there may be several candidates competing for the roles of $\underline{G}$ and $P^{\downharpoonleft}\&Q^{\downharpoonleft}$ and one has to select the "best ones". A preferred choice here is where the function $\underline{G}$ is the farthest from the most probable one. (This may be sometimes formulated as *minimization of a kind of an entropy.*)

Completion/Extrapolation of the Matrix G

You easily keep in you memory 100 000 words and you (subliminally) remember a few million occurrences of some pairs of them coming close together.[261] But, certainly, this never comes anywhere close to 10 000 000 000 – the number of entries in the matrix G: Most of G in your head is filled with question marks.

On the other hand, a couple of million examples is not so little when it comes to the matrix $\underline{G}$ that has only 90 000 entries, and if you have these, you replace $G(u,v) =?$ by $G(u,v) = G^{\downharpoonleft}(u,v) = \underline{G}(\underline{u},\underline{v})$.

As a result, when necessity arrives, you will not hesitate to accept or to reject as implausible pairs of relatively rare words coming next to each other, such as "intellectually posterior", "hydraulically superior", "superior posterior",

[260] If you check one pair of words per second – eight hours a day, five days a week – it will take more than 10 000 years to go through such a library.

[261] An average book contains about 100 000 words, and there are other sources of words besides books.

"intellectually hydraulically", "corticofugally inhibited", "intellectually candied", etc.[262]

Additive and Probabilistic Biclustering

Let all u in a set U carry *weights*, that are positive numbers denoted $|u|$. Then an *additive reduction* of such a *weighted space* is a map that adds up the weights. Namely, if, say, $P^{\downarrow}$ maps U onto some $\underline{U}$,

$$U \overset{P^{\downarrow}}{\to} \underline{U},$$

then this $\underline{U}$ is endowed with weights $|\underline{u}|$, that are sums of the weights of all u that go to $|\underline{u}|$,

$$|\underline{u}| = \sum_{P^{\downarrow}(u)=\underline{u}} |u|.$$

Similarly if the entries of a matrix (function) $G = G(u,v)$ are positive numbers, then the *additive reduction* $\underline{G}$ of G under

$$U \overset{P^{\downarrow}}{\to} \underline{U}, \quad \text{and} \quad V \overset{Q^{\downarrow}}{\to} \underline{V}$$

is where the weights of the entries from G add up to the corresponding entries in $\underline{G}$,

$$|G(\underline{u},\underline{v})| = \sum_{\substack{P^{\downarrow}(u)=\underline{v}, \\ Q^{\downarrow}(v)=\underline{v}}} |G(u,v)|.$$

Normalization and Probability. A positive weight function on a set U, $v \mapsto |v|$ is called *normalized* if these weights add up to one,

$$\sum_{u\in U} |u| = 1.$$

Then these weights $p_u = |u|$ are interpreted as "*probabilities of the events u*".

In general, e.g., when the weights represent frequencies of occurrence of u, one normalizes by setting

$$|u|_{\text{prob}} = \frac{|u|_{\text{freq}}}{\sum_{u\in U} |u|_{\text{freq}}},$$

thus turning $(U, |\dots|_{\text{freq}})$ to a *probability space* $(U, |\dots|_{\text{prob}})$.

Clusterisation of Letters: **vowels & consonants**

Additive probabilistic clustering is practical for small sets W, e.g., where A equals the set of 26 letters in the English alphabet, where the roughest such clustering divides the alphabet A into two classes, denote them $\circ$ and $\bullet$, according to relative frequencies of pairs of letters in English.

[262]In truth, you need several coclustering mechanisms working in parallel, e.g., to reject offhandedly "clusterizations scries" and the like.

The weights $|a|$ and $G(a,b)$ we use here are normalized frequencies of a and ab in a given text and we search for the (mathematically most natural) reduction, where the four weights

$$\begin{matrix} |\underline{G}(\circ,\circ)| & |\underline{G}(\circ,\bullet)| \\ |\underline{G}(\bullet,\circ)| & |\underline{G}(\bullet,\bullet)| \end{matrix}$$

of the entries of the 2×2 matrix $\underline{G}$ have *minimal relative entropy* with respect to the matrix of the products of the weights in $\underline{A} = \{\circ,\bullet\}$ that are

$$\begin{matrix} |\circ|\cdot|\circ| & |\circ|\cdot|\bullet| \\ |\bullet|\cdot|\circ| & |\bullet|\cdot|\bullet|. \end{matrix}$$

Most likely – this is certainly known, but I did not check it – this minimal entropy division of A into the classes $A_\circ$ and $A_\bullet$ coincides for the most part with the division of letters into vowels and consonants.

Graphically, G (approximately) "reduces" to the two vertex graph $\circ\!\!-\!\!\bullet$ by dividing the vertex set A into two classes/clusters

$$A = \boldsymbol{vo}wels\ \&\ \boldsymbol{co}nsonants$$

$$\overset{\boldsymbol{vo}}{\circ} - \overset{\boldsymbol{co}}{\bullet}.$$

Observe that this partition of A does not depend on any a priori knowledge of the "nature" of letters, but only on the relative frequencies of letters and pairs; the idea of *meaning* we attribute to these classes is not the source, but *a product* of the process of mathematical structuralisation where multiple biclusterisation plays an essential role.[263]

Geometrization of G

To be specific, let a function $G(u,v)$ in two variables take values in the set $\{0,1,2\}$ (standing for $nothing, something, much$) and let

$$U \overset{\mathcal{G}}{\to} \{0,1,2\}^V \text{ defined as } u \overset{\mathcal{G}}{\mapsto} f(v) = g_u(v) = G(u,v)$$

be the tautological map from the domain U of u to the space of $\{0,1,2\}$-valued functions $f(v)$ on the domain V of v, that is the Cartesian product space of copies of $\{0,1,2\}$ indexed by v from V,

$$\{0,1,2\}^V = \underbrace{\{0,1,2\}\times\{0,1,2\}\times\cdots\times\{0,1,2\}}_{V}.$$

For instance, let U be the set of 100 000 words and V a particular subset of say, 30–100 words selected by some preliminary mathematically defined process. For instance, these may be

- 100 most frequent words,

[263] A phonetically more accurate clustering needs tracking *triples* (quadruples?) of letters that will allow distinguishing certain pairs, such as "th" and "wh", for instance.

or, more interestingly,

★ 100 most frequent words from some class obtained by another biclustering algorithm,

such as

∘ a representative group of function words,
⁎ 100 most common verbs
✤ a list of 30 common four-legged animals
☞ a list of 30 common professions.

The space $\{0,1,2\}^V$ comes with many distance-like functions where a preferred one is the Hamming distance

$$\mathrm{dist}_V(f_1(v), f_2(v)) = \sum_{v\in V} |f_1(v) - f_2(v)|,$$

that passes to U via $\mathcal{G}$.

Then (bi)clustering of U according to possible coworkers v of u may be achieved by simple clustering relative to such a "distance" on U.

Reciprocal Clustering

If you choose a subset V of words in U at random, then there will be no preferred clustering of U for the distance coming from $dist_V$. On the other hand some exceptional V would lead to "clean" clusterings of U. Such special V are tight knit groups of similar words, such as, for instance: {*the*, *a*} or

{***Red***, ***Green***, ***Yellow***, ***Orange***, ***Purple***, ***Pink***, ***Brown***, ***Black***, ***Gray***, ***White***}.

Combinatorial Clustering: Why Pigs do not Fly

Let G be a graph on the vertex set V. Then the vertices v of this graph can be classified/clustered according to the combinatorics of subgraphs comprised of vertices and edges in the vicinity of v, where the simplest characteristic of such "vicinity" is the *valency of* v, that is, the number of edges attached to v.

Thus, for instance, one may first divide V into two parts V_{small}, where this valency is small, and V_{large}, where it is large, and then subdivide further according to the values of *pairs of numbers* of edges from v to V_{small} and to V_{large}.

Pigs, Pigeons, and Sparrows. "Birds fly" on 14 000 000 Google pages, "*Pigs fly*" on 3 000 000 pages, "*pigeons fly*" on 1 000 000 pages, and "sparrows fly" on 500 000 pages.

What distinguishes pigs from birds is not the sheer numbers of occurrences of sentences with "pig&fly" or "bird&fly" in them, but the numbers of combinatorial structures displayed by these sentences: There are by far less types of sentences with the words "*pig*" and "*fly*" in them, than of those with "*pigeon*" and "*sparrow*" replacing "*pig*".

This is similar to the use of prepositions in English, e.g., *under* and *in*, that may be accompanied by *different* kinds of nouns and/or verbs; yet the geometries/combinatorics of their vicinities in the "*network of short English sentences*" look, nevertheless, quite *similar* that brings all(?) prepositions to the same cluster.

On Biclustering Algorithms

Detection of clusters of "natural units" u in a U, e.g., of words, may not need the full knowledge of $G(u,v)$ at all (u,v) but only for v taken from special *small subsets* in V. For instance, "*the*" divides other words into two groups according to their systematic occurrence just before or just after "*the*".

This does not work for general G, and biclustering of sets, say, of cardinalities 100 000, associated to reductions $U \to \underline{U}$ and $V \to \underline{V}$ to sets $\underline{U}$ and $\underline{V}$ of cardinalities of order 300 seems, generically speaking, computationally unfeasible.

On the other hand there is a variety of heuristic algorithms that work pretty well for functions G coming from "life".

Words in Contexts: Biclustering and Triclustering

Biclustering may be applied to the function $G(w,x)$ where w are words and the variable x represents a context, e.g., a book from some collection X.

The natural function G encodes (frequent) presence/absence of a w in x and biclusterisation serves to classify books by topics according to their "key words", but the words themselves become classified by topics they are frequently used in, such as: *chemistry of plants, animal foods*, etc.

Structurally more informative classification, e.g., with an organisation of classes as *trees* with several branches, may be achieved with tri-clusterisation for $G(w_1,w_2,x)$ recording pairs of words (w_1,w_2) that appear in the same book x.

In general, however, it is unclear how to proceed with tri-clustering, partly because there is no convincing counterpart to the above "geometrization" $\mathcal{G}$ of G. On the other hand, most(?) multiple interactions appear as "combinations" of binary ones and multi-clustering reduces to several biclusterings.

Conclusion

Co-clustering is neither the final product of building a structure from "flows of words" nor is it an "atomic unit" of such a structure but rather a large molecule with simple, yet non-trivial, internal architecture where this molecule, in turn, serves as a building block for more elaborate syntactic structures.

The simplicity of this "mathematical molecule" makes it quite versatile: One can modify it in many ways and adjust it to building a variety of different global structures.

For instance: Reductions of $U\&V$ that lead to (approximate) clusterisations of U and/or V are (not always) composable and combinatorics of systems of (not quite) commutative diagrams of reductions represents an interesting "higher-order structure" in $U\&V$.

And besides mere classification a more subtle structure of a language may be extracted from a "distance" on V induced by the above $\mathcal{G}$ from a space of functions on some auxiliary set V, where the essential properties of such a "distance" are encoded by the (not-quite) *category* of approximate partial isometries of U with respect to this "distance".

One may continue indefinitely along these lines but one has to stop somewhere. Wings of imagination supplied by the power of mathematics can bring you beyond whatever can be reached by a more pedestrian kind of thinking. But if you fly too high in the sky of math you may miss your destination down on Earth.

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